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Measures of Growth and Approximation of Entire Harmonic Functions in n-Dimensional Space in Some Banach Spaces

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Abstract: The relationship between the classical order and type of an entire harmonic function in space $\mathbb{R}^n, n \geq 3$, and the rate of its best harmonic polynomial approximation for some Banach spaces of functions harmonic in the ball of radius R has been studied.

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1. Introduction

Several authors like Vakarchuk [24], Vakarchuk and Zhir [25,26], Srivastava and Kumar [20], Harfaoui [7] and others have studied the growth parameters of an entire function in terms of coefficients occurring in its Taylor series expansion and polynomial approximation errors in some Banach spaces. Since entire harmonic functions play an important role in physics and mechanics to describe different stationary processes and in mathematical research, it is significant to study the growth characteristics order and type of entire harmonic functions in terms of coefficients occurring in its Fourier-Laplace series [23] and harmonic polynomial approximation errors in space $\mathbb{R}^n, n \geq 3$ in some Banach spaces. To the best of our knowledge this study has not been done so far. In this paper our aim is to bridge this gap.
A number of papers [3,4,10-17,19,21] were devoted to establishing a relation between

the growth of entire harmonic functions in $\mathbb{R}^n, n \geq 3$ and the behavior of expansion coefficients, spherical harmonics and harmonic polynomial approximation errors. In particular, when we discuss time dependent problems in $\mathbb{R}^3$ it leads to study the harmonic functions in $\mathbb{R}^4$. Therefore, to study the entire harmonic functions in $\mathbb{R}^n, n \geq 3$ is reasonable.

Let $x \in \mathbb{R}^n (n \geq 3)$ be an arbitrary point where $x = (x_1, x_2, \dots, x_n)$ and put $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. The set of all non-constant entire harmonic functions on $\mathbb{R}^n$ is denoted by H. For each $u \in H, r > 0$, the Fourier-Laplace series expansion of u be given as [23]

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x; u) r^k,$$

where $x \in S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ a unit sphere in $\mathbb{R}^n$ centered at the origin and

$$\begin{aligned} Y^{(k)}(x; u) =& a_1^{(k)} Y_1^{(k)}(x) + a_2^{(k)} Y_2^{(k)}(x) + \cdots + a_{\gamma_k}^{(k)} Y_{\gamma_k}^{(k)}(x), \\ a_j^{(k)} &= (u, Y_j^{(k)}) = \frac{\Gamma(n/2)}{2(\pi)^{\frac{n}{2}}} \int_{S^n} u(x) Y_j^{(k)}(x) dS, j = \overline{1, \gamma_k}, \\ \gamma_k &= \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}. \end{aligned}$$

Here dS is the element of the surface area on the sphere S^n, $(u, Y_j^{(k)})$ is the scalor product in $L^2(S^n)$ and $Y^{(k)}$ is a spherical harmonic of degree $k, k \in Z_+ = \{0, 1, 2, \dots,\}$ on the unit sphere $S^n (n \geq 2)$ [22].

Let $B_R^n = \{y \in \mathbb{R}^n : |y| \leq R\}$ be the ball of radius R in space $\mathbb{R}^n, n \geq 3$ centered at the origin, and $\overline{B_R^n}$ be the closure of B_R^n. We denote H_R, the class of harmonic functions in B_R^n and continuous on $\overline{B_R^n}, 0 < R < \infty$.
We now consider some of the Banach spaces.

1. The space B of functions harmonic in the ball B_R^n and continuous on $\overline{B_R^n}$ i.e., $u \in H_R$ with norm $\|u\| = \max_{y \in \overline{B_R^n}} |u(y)| < \infty$.
2. The Hardy spaces $H_p, p \geq 1$, of functions harmonic in the ball B_R^n with norm
$$\|u\|_{H_p} = \sup_{0<r<R} M_p(r; u), M_p(r; u) = (\frac{1}{(2\pi)^n} \int_{T^n} |u(re^{it}x)|^p dt)^{\frac{1}{p}}, p \in [1, \infty),$$
where $T^n = \{x \in \mathbb{R}^n : 0 \leq x_j \leq 2\pi, j = \overline{1, n}\}$,
$\|u\| = \sup_{y \in B_R^n} |u(y)|, p = \infty$.
3. The Bergman spaces H_p' of functions harmonic in the ball B_R^n for $p \in [1, \infty)$ with the norm
$$\|u\|_{H_p'} = (\frac{1}{(\pi)^n} \int_{S^n} |u(rx)|^p dx)^{\frac{1}{p}}.$$

4. The spaces $A_p, p \in (0,1)$ of functions harmonic in the ball B_R^n with norm

$$\|u\|_{A_p} = \int_{S^n} (\frac{R-r}{R})^{\frac{1}{p}-2} M_1(r;u)dr.$$

5. The spaces $B_{p,q,\lambda}, 0 < p < q \leq \infty, \lambda > 0$, of functions harmonic in the ball B_R^n with the norm

$$\|u\|_{p,q,\lambda} = \{\int_{S^n} (\frac{R-r}{R})^{\lambda(\frac{1}{p}-\frac{1}{q})-1} M_q^{\lambda}(r;u)dr\}^{\frac{1}{\lambda}}, \lambda < \infty,$$

and

$$\|u\|_{p,q,\infty} = \sup_{0<r<R} \{(\frac{R-r}{R})^{(\frac{1}{p}-\frac{1}{q})} M_q(r;u)\}, \lambda = \infty,$$

for $\min(q,\lambda) \geq 1, B_{p,q,\lambda}$ are Banach spaces.

We denote a Banach space X formed by the functions harmonic in B_R^n with finite norm $\|.\|$ given by (1-5).
An approximation error of function $u \in H_R$ by harmonic polynomials $P \in \Pi_k$ is defined as

$$E_R^k(u) = \inf\{\|u(y) - P(y)\|, y \in \overline{B_R^n},$$

where Π_k be a set of harmonic polynomials of degree not exceeding k.
The relationship between the order and type of an entire function f in terms of the sequence $E_R^k(f)$ in the space H_2' were obtained in [19] and for the spaces $H_p', p \geq 1$ were studied by Ibragimov and Shikhaliev [8,9]. The spaces $A_p, p \in (0,1)$ of functions analytic in the unit disk were first studied by Hardy and Littlewood [6] and later by Romberg, Duren and Shields [2]. The spaces $B_{p,q,\lambda}, 0 < p < q \leq \infty, \lambda > 0$ were considered in [5,6]. The order and type of entire functions in terms of approximation errors $E_R^k(f)$ in the spaces $B_{p,q,\lambda}$ were obtained by Vakarchuk [24].

2. Auxiliary Results

Lemma 2.1. *Let $u \in X$ and $u(\tau x) = \sum_{k=0}^{\infty} Y^k(x;u)\tau^k, 0 < \tau < R$, be an entire harmonic function in $\mathbb{R}^n$. Then*

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_X}{R^k}\}^{\frac{1}{k}} = 1. \tag{2.1}$$

Proof. For an entire harmonic function u in the space B and $H_p, 0 < p \leq \infty$, respectively, the quantity $\|\tau^k\|_X, k \in \mathbb{Z}_+$ is

$$\|\tau^k\|_X = R^k$$

it gives (2.1).
In the space $X = H'_p, p \geq 1$, we have

$$\frac{\|\tau^k\|_{H'_p}}{R^k} = (kp+2)^{\frac{1}{p^2}} \leq \{kp(1+\frac{2}{kp})\}^{\frac{1}{p^2}}, k \geq 0 \tag{2.2}$$

or

$$\frac{\|\tau^k\|_{H'_p}}{R^k} \leq \chi_{H'_p} k^{\frac{1}{p^2}} \tag{2.3}$$

where $\chi_{H'_p} = p^{\frac{1}{p^2}}(1+\frac{1}{p})^{\frac{1}{p^2}}$.
From (2.3) we obtain the following upper bound

$$\lim_{k\to\infty}\{\frac{\|\tau^k\|_{H'_p}}{R^k}\}^{\frac{1}{k}} \leq 1. \tag{2.4}$$

For lower bound using (2.2) and we get

$$\frac{\|\tau^k\|_{H'_p}}{R^k} \geq p^{\frac{1}{p^2}} k^{\frac{1}{p^2}} \geq (pk)^{\frac{1}{p^2}}$$

or

$$\lim_{k\to\infty}\{\frac{\|\tau^k\|_{H'_p}}{R^k}\}^{\frac{1}{k}} \geq 1. \tag{2.5}$$

Combining (2.4) and (2.5) we get the required result.
In the space $X = A_p, 0 < p < 1$, we have

$$\frac{\|\tau^k\|_{A_p}}{R^k} = (2\pi)^{-\frac{1}{p}}(B(kp+1;\frac{1}{p}-1))^{-\frac{1}{p^2}}. \tag{2.6}$$

The right hand side of (2.6) can be estimated by using the relation between the Euler integral of the first kind $B(a,b)$ and Γ function for $a, b > 0$,

$$B(a,b) = \frac{\Gamma a \Gamma b}{\Gamma(a+b)}$$

and the asymptotic relation

$$\frac{\Gamma(\xi+s_1)}{\Gamma(\xi+s_2)} = \xi^{s_1-s_2}(1+\frac{(s_1-s_2)(s_1+s_2-1)}{2x} + o(|\xi^{-2}|)$$

where $|\xi| \geq 1, \xi \in \mathbb{R}$ and s_1 and s_2 are arbitrary fixed real numbers.
Set $\xi = kp, s_1 = \frac{1}{p}$ and $s_2 = 1$, for sufficiently large k, for $k \geq 1$ in above relations, we get

$$\begin{aligned}\frac{\|\tau^k\|_{A_p}}{R^k} &= [\frac{(2\pi)^{-p}\Gamma(kp+\frac{1}{p})}{\Gamma^{\frac{1}{p^2}}(\frac{1}{p}-1)(kp+1)}]^{\frac{1}{p^2}} \\ &= \frac{(2\pi)^{-p}p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)}(k(1+\frac{(\frac{1}{p}-1)\frac{1}{p}}{2kp} + O(k^{-2}p^2)))^{\frac{1}{p^2}} \\ &\leq \chi_{A_p} k^{\frac{1}{p^2}},\end{aligned}$$

where

$$\chi_{A_p} = \frac{(2\pi)^{-p} p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)} (1 + (\frac{1}{p}-1)\frac{1}{p} + A)^{\frac{1}{p^2}}$$

here A is an absolute constant independent of k.

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{A_p}}{R^k}\}^{\frac{1}{k}} \leq 1. \tag{2.7}$$

For lower bound we have

$$\frac{\|\tau^k\|_{A_p}}{R^k} \geq \frac{(2\pi)^{-p} p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)} k^{\frac{1}{p^2}} \geq \{\frac{k}{\Gamma(\frac{1}{p}-1)}\}^{\frac{1}{p^2}}$$

or

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{A_p}}{R^k}\}^{\frac{1}{k}} \geq 1. \tag{2.8}$$

Using (2.7) and (2.8) we get the required result.
Following on the lines of [26] for single complex variable we obtain for the space $X = B_{p,q,\lambda}, 0 < p < q \leq \infty, 0 < \lambda \leq \infty$ that

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{B_{p,q,\lambda}}}{R^k}\}^{\frac{1}{k}} = 1.$$

□

Lemma 2.2. *Let $u \in X$ and let*

$$u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)\tau^k \quad in \quad space \quad \mathbb{R}^n, n \geq 3, 0 < \tau < R.$$

Then

$$|Y^{(k)}(x; u)| \|\tau^k\|_X \leq \frac{2\sqrt{2}(k + 2\nu)}{C\sqrt{(2\nu)!}(2\nu + 1)(k - 1 + 2\nu)^{2\nu}} (\frac{r}{R})^{k-1} E_R^{k-1}(u) \leq \|u(\tau x)\|_X,$$

where C is a constant independent of u and τx.

Proof. Using the addition theorem [1] for the Gegenbauer polynomials C_k^ν of degree k and order ν, we have

$$\int_{S^n} C_k^\nu[(x, \zeta)] P(\tau\zeta) dS(\zeta) = 0$$

where $P \in \Pi_{k-1}, 0 < \tau < R, x \in S^n, \zeta \in S^n$, and

$$Y^{(k)}(x; u) r^k = \frac{2(k + \nu)}{d_n w_n} \int_{S^n} C_k^\nu[(x, y)] u(ry) dS(y), \tag{2.9}$$

where $k \in Z, d_2 = 1, d_n = n-2$ at $n > 2, \nu = \frac{n-2}{2}, w_n = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.
Rewrite (2.9) as

$$Y^{(k)}(x;u)\tau^k = \frac{(k+\nu)}{\nu w_n}\int_{S^n} C_k^{\nu}[(x,\zeta)][u(\tau\zeta) - P(\tau\zeta)]dS(\zeta). \tag{2.10}$$

Since $\max_{-1\le t\le 1}|C_k^{\nu}(t)| = C_k^{\nu}(1)$, from [1] we have $C_k^{\nu}(1) = \frac{(k+2\nu-1)!}{(d_n-1)!k!}$ and from (2.10) we obtain

$$\begin{aligned}|Y^{(k)}(x;u)|\|\tau^k\|_X \le & \frac{(k+\nu)}{\nu w_n}\|u(\tau\zeta) - P(\tau\zeta)\|_X C_k^{\nu}(1) w_n \\ \le & \frac{2(k+2\nu)^{2\nu}}{(2\nu)!}\|u(\tau\zeta) - P(\tau\zeta)\|_X\end{aligned}$$

there exists a polynomial $P^* \in \Pi_{k-1}$ for which

$$\|u(\tau\zeta) - P(\tau\zeta)\|_X \le C\max|u(\tau\zeta) - P^*(\tau\zeta)| \le 2E_R^{k-1}(u).$$

So we have

$$|Y^{(k)}(x;u)|\|\tau^k\|_X \le \frac{4C(k+2\nu)^{2\nu}}{(2\nu)!}E_R^{k-1}(u). \tag{2.11}$$

Now consider $Q(\tau\zeta) = \sum_{j=0}^{k} Y^{(j)}(\zeta;u)\tau^j$, since $Q \in \Pi_k$, we have

$$E_R^k(u) \le \|u(\tau\zeta) - Q(\tau\zeta)\|_X \le C\max_{\tau\zeta\in\overline{B_R^n}}|u(\tau\zeta) - Q(\tau\zeta)|,$$

Using a result of [27] we have

$$\begin{aligned}E_R^k(u) \le & \sum_{j=k+1}^{\infty} C\max_{\zeta\in S^n}|Y^{(j)}(\zeta,u)|R^j \\ \le & C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}(\frac{R}{r})^j \\ = & C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}(\frac{R}{r})^{j-k}.\end{aligned} \tag{2.12}$$

For $r > eR$, the maximum value of last sum can be estimate as

$$\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}(\frac{R}{r})^{j-k} \le e^k\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}e^{-j} \le e^k\int_k^{\infty}(t+2\nu)^{2\nu}e^{-t}dt.$$

Set $\eta = 2\nu, \theta_\eta(t) = (t+\eta)^\eta$ and integrating $(\eta+1)$ times by parts, we get

$$\int_k^\infty \theta_\eta(t)e^{-t}dt = [-e^{-t}(\theta_\eta(t) + \theta'_\eta(t) + \cdots + \theta_\eta^{(n)}(t))]|_k^\infty$$
$$= e^{-k}\sum_{i=0}^{\eta}\frac{\eta!(k+\eta)^{\eta-i}}{(\eta-i)!}, \quad since \ \ \theta_\eta^{(i)} = \frac{\eta!}{(\eta-i)}(t+\eta)^{\eta-i}, i = \overline{1,\eta},$$
$$= e^{-k}\sum_{i=0}^{2\nu}\frac{(2\nu)!(k+2\nu)^{2\nu-1}}{(2\nu-i)!}.$$

The maximum value of above term is $(2\nu+1)!(k+2\nu)^{2\nu}$. Hence from (2.12) we have

$$E_R^k(u) \le C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k(2\nu+1)(k+2\nu)^{2\nu}. \tag{2.13}$$

Combining (2.11) with (2.13) we get

$$|Y^{(k)}(x;u)|\|\tau^k\|_X \le \frac{4(k+2\nu)^{2\nu}}{(2\nu)!}E_R^{k-1}(u) \le C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k(2\nu+1)(k+2\nu)^{2\nu}, \tag{2.14}$$

above inequality (2.14) gives the required result. □

Lemma 2.3. *Let*

$$\alpha_1 = \liminf_{k\to\infty}(\|\tau^k\|)^{\frac{1}{k}} \quad and \quad \alpha_2 = \limsup_{k\to\infty}(\|\tau^k\|)^{\frac{1}{k}}.$$

Then $\alpha_1 \ge R$ and $\alpha_2 < \infty$.

Proof. Suppose $\beta_k = (\|\tau^k\|)^{\frac{1}{k}}$. First we prove that $\alpha_2 < \infty$. On the contrary we assume that there exists a subsequence β_{k_m} such that $\lim_{m\to\infty}\beta_{k_m} = \infty$. Consider a function u_0 such that

$$u_0(\tau x) = \sum_{m=0}^{\infty}(\beta_{k_m})^{-\frac{k_m}{2}}\tau^{k_m}.$$

The function $u_0(\tau x)$ is entire and therefore belong to X. However, in this case using Lemma 2.2, for any $m \in \mathbb{N}$, we get

$$(\beta_{k_m})^{-\frac{k_m}{2}}\|\tau^{k_m}\| \le \|u_0(\tau x)\| < \infty,$$

which is impossible. Hence $\alpha_2 < \infty$. Now in order to prove $\alpha_1 \ge R$ we assume that $\alpha_1 < R$. Set $\delta \in (\alpha_1; R)$ and consider a function

$$u_0(\tau x) = \sum_{m=0}^{\infty}\delta^{-k_m}\tau^{k_m} \tag{2.15}$$

where k_m is a sequence such that $\liminf_{k\to\infty} \beta_k = \lim_{m\to\infty} \beta_{k_m} = \alpha_1$. The function $u_0(\tau x)$ is harmonic in the ball $B^n_\delta(u_0), \delta < R$ but not harmonic in ball B^n_R. It is clear that a sequence of partial sums $S_{k,u_0}(\tau x)$ of series (2.15) is fundamental in the Banach space X and, therefore, convergence in it to a function $u_1(\tau x) \in X$. We now prove that the Fourier-Laplace coefficients of the functions $u_0(\tau x)$ and $u_1(\tau x)$ are same. For fixed $m \in \mathbb{N} \cup \{0\}, k > m$, we have

$$Y^{(k)}(x; u_1) = Y^{(k)}(x; S_{k,u_0}) + Y^{(k)}(x; u_1 - S_{k,u_0}) = Y^{(k)}(x; u_0) + Y^{(k)}(u_1 - S_{k,u_0}).$$

Proceeding to the limit as $k \to \infty$ and using Lemma 2.2, it gives $Y^{(k)}(x; u_1) = Y^{(k)}(x; u_0)$. Hence the function $u_1(\tau x) \in X$ but not harmonic in B^n_R, which contradicts the property of the space X. Hence $\alpha_1 \geq R$. □

Lemma 2.4. *Let $u \in X$ and let K be a compact subset of $\mathbb{R}^n, n \geq 3, K \subset B^n_R$. Then, for $\tau x \in K$,*

$$|u(\tau x)| \leq C\|u(\tau x)\|.$$

Proof. Let $\gamma = \sup\{|\tau x| : \tau x \in K\}, \gamma < R$. Now write the expansion of u in the Fourier-Laplace series and estimates its modulus by using Lemma 2.2, we get

$$u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)\tau^k,$$

$$|u(\tau x)| \leq \sum_{k=0}^{\infty} |Y^{(k)}(x; u)||\tau^k| \leq \|u(\tau x)\| \sum_{k=0}^{\infty} \frac{\gamma^k}{\|\tau^k\|} \leq C\|u(\tau x)\|$$

as the series is convergent by Lemma 2.3. □

3. Main Results

Theorem 3.1. *Let $u \in X$. The condition*

$$\lim_{k\to\infty} (E^k_R(u))^{\frac{1}{k}} = 0$$

is necessary and sufficient for the function u to be entire.

Proof. Let $u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)\tau^k$ in space $\mathbb{R}^n, n \geq 3, 0 < \tau < R$. In view of Lemma 2.2,

$$|Y^{(k)}(x; u)|\|\tau^k\|_X \leq \frac{2\sqrt{2}(k + 2\nu)}{C\sqrt{(2\nu)!}(2\nu + 1)(k - 1 + 2\nu)^{2\nu}} (\frac{r}{R})^{k-1} E^{k-1}_R(u)$$

it gives

$$\lim_{k\to\infty} |Y^{(k)}(x; u)|^{\frac{1}{k}} \leq \lim_{k\to\infty} (\frac{2\sqrt{2}(k + 2\nu)}{C\sqrt{(2\nu)!}(2\nu + 1)(k - 1 + 2\nu)^{2\nu}} (\frac{r}{R})^{k-1} \frac{E^{k-1}_R(u)}{\|\tau^k\|_X})^{\frac{1}{k}} = 0,$$

therefore the function u is entire. Now for necessity, from Lemma 2.3, we have

$$\frac{E_R^{k-1}(u)R^k}{\|\tau^k\|_X} \leq \frac{\sqrt{(2\nu)!}(2\nu+1)(k-1+2\nu)\|u(\tau x)\|_X R^k r}{2\sqrt{2}(k+2\nu)\|\tau^k\|_X r^k}.$$

Since u is entire and $u \in X$ for any $r > 1$, we have

$$0 \leq \lim_{k\to\infty} (\frac{E_R^{k-1}(u)R^k}{\|\tau^k\|_X})^{\frac{1}{k}} \leq \frac{1}{r} \limsup_{k\to\infty} (\frac{\sqrt{(2\nu)!}(2\nu+1)(k-1+2\nu)\|u(\tau x)\|_X R^k r}{2\sqrt{2}(k+2\nu)\|\tau^k\|_X})^{\frac{1}{k}} \leq \frac{1}{r}.$$

Now by arbitrariness of $r > 1$, we obtain

$$\lim_{k\to\infty} (E_R^k(u))^{\frac{1}{k}} = 0.$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *For a function $u \in X$ to be an entire harmonic function in space $\mathbb{R}^n, n \geq 3$, of finite order $0 < \rho < \infty$, the necessary and sufficient condition is*

$$\limsup_{k\to\infty} \frac{k\log k}{\log(\frac{\|\tau^k\|}{E_R^k(u)})} = \rho. \tag{3.1}$$

Proof. To prove sufficiency part let (3.1) holds therefore the condition of Theorem 3.1 is satisfied and, hence, the function u is entire harmonic in space $\mathbb{R}^n, n \geq 3$ and we denote its order by ρ_1. Thus, on account of Lemma 2.2 we obtain

$$\rho_1 = \limsup_{k\to\infty} \frac{k\log k}{\log|Y^{(k)}(x;u)|} \leq \limsup_{k\to\infty} \frac{k\log k}{\log(\frac{\|\tau^k\|}{E_R^k(u)})} = \rho. \tag{3.2}$$

According to the condition of theorem we have to show that $\rho_1 > 0$. On the contrary, we assume that

$$\limsup_{k\to\infty} \frac{k\log k}{\log|Y^{(k)}(x;u)|} = 0.$$

Then for any $\varepsilon, 0 < \varepsilon < R$, there exists K_ε such that, for $k > K_\varepsilon$

$$k\log k < -\varepsilon \log|Y^{(k)}(x;u)|$$

or

$$|Y^{(k)}(x;u)| < k^{-\frac{k}{\varepsilon}}.$$

Now for sufficiently large K_ε we have

$$\|\tau^k\|_X \leq (\mu_2+\varepsilon)^k \quad \text{and} \quad \|\tau^k\|_X \geq (R-\varepsilon)^k \quad \text{for} \quad k \geq K_\varepsilon.$$

Then

$$E_R^k(u) \leq \| \sum_{j=k+1}^{\infty} |Y^{(k)}(x;u)|\tau^k\|_X \leq \sum_{j=k+1}^{\infty} j^{-\frac{j}{\varepsilon}}(\mu_2+\varepsilon)^j$$
$$\leq \sum_{j=k+1}^{\infty} (k+1)^{-\frac{j}{\varepsilon}}(\mu_2+\varepsilon)^j = (k+1)^{-\frac{(k+1)}{\varepsilon}}(\mu_2+\varepsilon)^{k+1}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\varepsilon}}})^{-1}. \tag{3.3}$$

We assume $k+1 \geq (\mu_2+\varepsilon)^{\varepsilon}$ in (3.3), it gives

$$\frac{\|\tau^k\|_X}{E_R^k(u)} \geq (\frac{R-\varepsilon}{\mu_2+\varepsilon})^{k+1}(k+1)^{\frac{(k+1)}{\varepsilon}}(1-\frac{\mu_2+\varepsilon}{(k+1)^{\frac{1}{\varepsilon}}}),$$

or

$$\log(\frac{\|\tau^k\|_X}{E_R^k(u)})^{\frac{1}{k}} \geq (\frac{k+1}{k})\log(\frac{R-\varepsilon}{\mu_2+\varepsilon}) + \frac{k+1}{k\varepsilon}\log(k+1) + \frac{1}{k}\log(1-\frac{\mu_2+\varepsilon}{(k+1)^{\frac{1}{\varepsilon}}})$$

or

$$\liminf_{k\to\infty} \frac{\log(\frac{\|\tau^k\|_X}{E_R^k(u)})^{\frac{1}{k}}}{\log k} \geq \frac{1}{\varepsilon},$$

or

$$\rho = \limsup_{k\to\infty} \frac{k\log k}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}} \leq \varepsilon,$$

which contradicts our assumption. Now we consider the case for $\varepsilon \in (0, \frac{R}{2}) \cap (0, \rho_1)$. From the left hand side of (3.2) we conclude that there exists $K_\varepsilon \in \mathbb{N}(\varepsilon)$ such that

$$|Y^{(k)}(x;u)| < k^{-\frac{k}{(\rho_1+\varepsilon)}}$$

for all $k > K_\varepsilon$. Let K_ε be sufficiently large such that $\|\tau^k\|_X \leq (\mu_2+\varepsilon)$ and $\|\tau^k\| \geq (R-\varepsilon)^k$ for $k \geq K_\varepsilon$. Then for $k > K_\varepsilon$,

$$E_R^k(u) \leq \| \sum_{j=k+1}^{\infty} Y^{(j)}(x;u)\tau^j\| \leq \sum_{j=k+1}^{\infty} |Y^{(j)}(x;u)|\|\tau^k\|$$
$$\leq \sum_{j=k+1}^{\infty} (j)^{-\frac{j}{\rho_1+\varepsilon}}\|\tau^j\| \leq \sum_{j=k+1}^{\infty} (k+1)^{-\frac{j}{\rho_1+\varepsilon}}(\mu_2+\varepsilon)^j \tag{3.4}$$
$$= \frac{(\mu_2+\varepsilon)^{k+1}}{(k+1)^{\frac{(k+1)}{(\rho_1+\varepsilon)}}}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}})^{-1},$$

or

$$\frac{\|\tau^k\|(k+1)^{\frac{1}{\rho_1+\varepsilon}}}{E_R^k(u)} \geq \frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{k+1}}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}}),$$

or

$$\rho_1+\varepsilon \geq \frac{(k+1)\log(k+1)}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}}\Big(1+\frac{(\rho_1+\varepsilon)}{(k+1)\log(k+1)}\log(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}})+ \frac{\rho_1+\varepsilon}{(k+1)\log(k+1)}\log\frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{(k+1)}}\Big). \tag{3.5}$$

Proceeding to limit as $k\to\infty$, we get $\rho_1+\varepsilon\geq\rho$. Since ε is arbitrary this implies that $\rho_1\geq\rho$. In view of (3.2), we get $\rho_1=\rho$, hence the sufficient part is completed. In order to prove the necessary part we assume that $u\in X$ be an entire harmonic function of finite order ρ, i.e.,

$$\limsup_{k\to\infty}\frac{k\log k}{-\log|Y^{(k)}(x;u)|}=\rho.$$

Set

$$\rho_1=\limsup_{k\to\infty}\frac{k\log k}{\log\frac{\|\tau^k\|_X}{E_R^k(u)}}.$$

Here ρ_1 and ρ are interchanged as compared with the proof of sufficiency part and show that $\rho_1=\rho$. By analogy with (3.2), Lemma 2.2 gives $\rho_1\geq\rho$. Following the same fact as in the sufficiency part, we can say that, for any $\varepsilon, 0<\varepsilon<R$ there exists K_ε such that

$$|Y^{(k)}(x;u)|<k^{-\frac{k}{(\rho+\varepsilon)}} \quad and \quad (R-\varepsilon)^k\leq\|\tau^k\|\leq(\mu_2+\varepsilon)^k$$

for $k>K_\varepsilon$. Following (3.3) and (3.4) (with ρ_1 and ρ interchanged), we get

$$\rho+\varepsilon \geq \frac{(k+1)\log(k+1)}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}}\Big(1+\frac{(\rho+\varepsilon)}{(k+1)\log(k+1)}\log(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho+\varepsilon}}})+ \frac{\rho+\varepsilon}{(k+1)\log(k+1)}\log\frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{(k+1)}}\Big).$$

Proceeding to limit as $k\to\infty$, it gives $\rho\geq\rho_1$. Hence the proof is completed. □

Theorem 3.3. For a function $u\in X$ to be an entire harmonic function of finite order $\rho\in(0,\infty)$ and normal type $\sigma\in(0,\infty)$, the necessary and sufficient condition is that

$$\limsup_{k\to\infty}\frac{k}{e\rho}\Big(\frac{E_R^k(u)}{\|\tau^k\|}\Big)^{\frac{\rho}{k}}=\sigma. \tag{3.6}$$

Proof. In order to prove sufficiency we assume that $u\in X$ and satisfies the condition of Theorem 3.3 with some positive ρ and σ. Then (3.1) follows from (3.6), therefore, u is an entire harmonic function of order ρ. Assume that the type of u is T. We have to prove $T=\sigma$. Now using the classical coefficient formula for the type of an entire harmonic function $u\in X$

$$T=\limsup_{k\to\infty}\frac{k}{e\rho}|Y^{(k)}(x;u)|^{\frac{\rho}{k}} \tag{3.7}$$

with Lemma 2.3, we obtain $T\le\sigma$. To prove the reverse inequality we have from (3.7) that for any $\varepsilon>0$, there exists $K_\varepsilon\in\mathbb{N}$ such that, for $k>K_\varepsilon$,

$$|Y^{(k)}(x;u)|<(\frac{\rho e(T+\varepsilon)}{k})^{\frac{k}{\rho}}. \tag{3.8}$$

Following the same techniques as (3.4) and (3.5), we have from (3.8),

$$E_R^k(u)\le\sum_{j=k+1}^{\infty}(\frac{\rho e(T+\varepsilon)}{j})^{\frac{j}{\rho}}\|\tau^j\|\le(\frac{\rho e(T+\varepsilon)}{k+1})^{\frac{k+1}{\rho}}(\mu+\varepsilon)^{(k+1)}(1-\frac{C^*}{(k+1)^{\frac{1}{\rho}}})^{-1} \tag{3.9}$$

where $C^*=(\mu+\varepsilon)(\rho e(T+\varepsilon))^{\frac{1}{\rho}}$. Now in view of (3.9), we obtain

$$T+\varepsilon\ge\frac{(k+1)}{e\rho}(\frac{E_R^k(u)}{\|\tau^k\|})^{\frac{\rho}{(k+1)}}\frac{\|\tau^k\|^{\frac{\rho}{(k+1)}}}{(\mu+\varepsilon)^\rho}(1-\frac{C^*}{(k+1)^{\frac{1}{\rho}}})^{\frac{\rho}{(k+1)}}.$$

Proceeding the limit sup as $k\to\infty$, we get

$$T+\varepsilon\ge\sigma(\frac{\mu}{\mu+\varepsilon})^\rho.$$

Since ε is arbitrary and approaches to zero, we get $T\ge\sigma$. Hence the sufficiency part is completed.
Now to prove necessity assume that $u\in X$ is an entire harmonic function of finite order and normal type. We denote its order and type as ρ and T respectively. Further, we have to show that $T=\sigma$. By virtue of (3.7) and Lemma 2.3, we obtain $T\le\sigma$. Finally, to prove $\sigma\le T$ repeat the reasoning of sufficiency part. This completes the proof of necessary part. Hence the proof of theorem is completed.

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