

Partial sums of a certain harmonic univalent meromorphic functions

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ABSTRACT: In the present paper we determine sharp lower bounds of the real part of the ratios of harmonic univalent meromorphic functions to their sequences of partial sums.

Let Σ_H denote the class of functions f that are harmonic univalent and sense-preserving in $U^* = \{z : |z| > 1\}$ which are of the form

$$f(z) = h(z) + \overline{g(z)},$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}.$$

Now, we define the sequences of partial sums of functions f of the form

$$\begin{aligned} f_s(z) &= z + \sum_{n=1}^s a_n z^{-n} + \overline{g(z)}, \\ \tilde{f}_r(z) &= g(z) + \sum_{n=1}^r \overline{b_n z^{-n}}, \\ f_{s,r}(z) &= z + \sum_{n=1}^s a_n z^{-n} + \sum_{n=1}^r \overline{b_n z^{-n}}. \end{aligned}$$

In the present paper we will determine sharp lower bounds for $\operatorname{Re} \left\{ \frac{f(z)}{f_s(z)} \right\}$, $\operatorname{Re} \left\{ \frac{f_s(z)}{f(z)} \right\}$, $\operatorname{Re} \left\{ \frac{f(z)}{\tilde{f}_r(z)} \right\}$, $\operatorname{Re} \left\{ \frac{\tilde{f}_r(z)}{f(z)} \right\}$, $\operatorname{Re} \left\{ \frac{f(z)}{f_{s,r}(z)} \right\}$, $\operatorname{Re} \left\{ \frac{f_{s,r}(z)}{f(z)} \right\}$.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D . It was shown by Clunie and Sheil-Small [4] that such harmonic function can be represented by $f = h + \bar{g}$, where h and g are analytic in D . Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$. There are numerous papers on univalent harmonic functions defined in a domain $U = \{z \in \mathbb{C} : |z| < 1\}$ (see [6,7], [14] and [15]). Hergartner and Schober [10] investigated functions harmonic in the exterior of the unit disc i.e $U^* = \{z \in \mathbb{C} : |z| > 1\}$. They showed that a complex valued, harmonic, sense preserving univalent function f , defined on U^* and satisfying $f(\infty) = \infty$ must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \quad (A \in \mathbb{C}), \quad (1.1)$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \quad (z \in U^*, \quad 0 \leq |\beta| < |\alpha|), \quad (1.2)$$

and $a = \bar{f}_z/f_z$ is analytic and satisfy $|a(z)| < 1$ for $z \in U^*$.

Let us denote by Σ_H the class of functions f that are harmonic univalent and sense-preserving in U^* , which are of the form

$$f(z) = h(z) + \overline{g(z)} \quad (z \in U^*), \quad (1.3)$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}.$$

Now, we introduce a class $\Sigma_H(c_n, d_n, \delta)$ consisting of functions of the form (1.3) such that

$$\sum_{n=1}^{\infty} c_n |a_n| + \sum_{n=1}^{\infty} d_n |b_n| < \delta \quad (d_n \geq c_n \geq c_2 > 0; \delta > 0). \quad (1.4)$$

It is easy to see that various subclasses of Σ_H consisting of functions $f(z)$ of the form (1.3) can be represented as $\Sigma_H(c_n, d_n, \delta)$ for suitable choices of c_n, d_n and δ studies earlier by various authors.

- $\Sigma_H(n, n, 1) = H_0^*$ (see Jahangiri and Silverman. [8]);
- $\Sigma_H(n + \gamma, n - \gamma, 1 - \gamma) = \Sigma_H^*(\gamma)$ ($0 \leq \gamma < 1, n \geq 1$) (see Jahangiri [5]);

- $\Sigma_H(|(n+1)\lambda - 1|, |(n-1)\lambda + 1|, 1 - \alpha) = \Sigma_H R(\alpha, \lambda) (0 \leq \alpha < 1, \lambda \geq 0, n \geq 1)$ (see Ahuja and Jahangiri [1]);
- $\Sigma_H(n + \alpha - \alpha\lambda(n+1), n - \alpha - \alpha\lambda(n-1), 1 - \alpha) = \Sigma_H S^*(\alpha, \lambda) (0 \leq \alpha < 1, 0 \leq \lambda \leq 1, n \geq 1)$ (see Janteng and Halim [9]),
- $\Sigma_H(n(n+2)^m, n(n-2)^m, 1) = MH^*(m) (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}, n \geq 1)$ (see Bostanci and Ozturk [2]);
- $\Sigma_H((n+\gamma)(n+2)^m, (n-\gamma)(n-2)^m, 1 - \gamma) = MH^*(m, \gamma) (0 \leq \gamma < 1, m \in \mathbb{N}_0, n \geq 1)$ (see Bostanci and Ozturk [3]).

Silvia [17] studied the partial sums of the convex functions of order α , later on Silverman [16] studied partial sum for starlike and convex functions. Very recently, Porwal [12], Porwal and Dixit [13] and Porwal [11] studied analogues interesting results on the partial sums of certain harmonic univalent functions.

Since to a certain extent the work in the harmonic univalent meromorphic functions case has paralleled that of the harmonic analytic univalent case, one is tempted to search results analogous to those of Porwal [11] for meromorphic harmonic univalent functions in U^* .

Now, we define the sequences of partial sums of functions f of the form (1.3) by

$$\begin{aligned} f_s(z) &= z + \sum_{n=1}^s a_n z^{-n} + \sum_{n=1}^{\infty} \overline{b_n z^{-n}}, \\ \tilde{f}_r(z) &= z + \sum_{n=1}^{\infty} a_n z^{-n} + \sum_{n=1}^r \overline{b_n z^{-n}}, \\ f_{s,r}(z) &= z + \sum_{n=1}^s a_n z^{-n} + \sum_{n=1}^r \overline{b_n z^{-n}}, \end{aligned} \tag{1.5}$$

when the coefficients of f are sufficiently small to satisfy the condition (1.4).

In the present paper, motivated essentially by the work of Silverman [16] and Porwal [11], we will determine sharp lower bounds for $\operatorname{Re} \left\{ \frac{f(z)}{f_s(z)} \right\}$,

$$\operatorname{Re} \left\{ \frac{f_s(z)}{f(z)} \right\}, \operatorname{Re} \left\{ \frac{f(z)}{\tilde{f}_r(z)} \right\}, \operatorname{Re} \left\{ \frac{\tilde{f}_r(z)}{f(z)} \right\}, \operatorname{Re} \left\{ \frac{f(z)}{f_{s,r}(z)} \right\} \text{ and } \operatorname{Re} \left\{ \frac{f_{s,r}(z)}{f(z)} \right\}.$$

2 Main Results

Theorem 1. *Let $s \in \mathbb{N}$ and let $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. Then*

$$(i) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_s(z)} \right\} > 1 - \frac{\delta}{c_{s+1}} \quad (z \in U), \tag{2.1}$$

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{f_s(z)}{f(z)} \right\} > \frac{c_{s+1}}{\delta + c_{s+1}} \quad (z \in U), \tag{2.2}$$

whenever

$$c_n \geq \begin{cases} \delta, & n = 2, 3, \dots, s, \\ c_{s+1}, & n = s + 1, s + 2, \dots \end{cases} \quad (2.3)$$

The estimates in (2.1) and (2.2) are sharp for the function given by

$$f(z) = z + \frac{\delta}{c_{s+1}} z^{-s-1} \quad (z \in U^*). \quad (2.4)$$

Proof. (i) To obtain the sharp lower bound given by (2.1), let us put

$$\begin{aligned} g_1(z) &= \frac{c_{s+1}}{\delta} \left\{ \frac{f(z)}{f_s(z)} - \left(1 - \frac{\delta}{c_{s+1}}\right) \right\} \\ &= 1 + \frac{\frac{c_{s+1}}{\delta} \sum_{n=s+1}^{\infty} a_n z^{-n}}{z + \sum_{n=1}^s a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^{-n}}. \end{aligned} \quad (2.5)$$

Then, it is sufficient to show that $\operatorname{Re} g_1(z) > 0$ ($z \in U^*$) or equivalently

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq 1 \quad (z \in U^*).$$

Since

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{\frac{c_{s+1}}{\delta} \sum_{n=s+1}^{\infty} |a_n|}{2 - 2 \left(\sum_{n=1}^s |a_n| + \sum_{n=1}^{\infty} |b_n| \right) - \frac{c_{s+1}}{\delta} \sum_{n=s+1}^{\infty} |a_n|}, \quad (2.6)$$

the last expression is bounded above by 1, if and only if

$$\sum_{n=1}^s |a_n| + \sum_{n=1}^{\infty} |b_n| + \frac{c_{s+1}}{\delta} \sum_{n=s+1}^{\infty} |a_n| \leq 1. \quad (2.7)$$

Then, it is sufficient to show that L.H.S. of (2.7) is bounded above by

$$\sum_{n=1}^{\infty} \frac{c_n}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n}{\delta} |b_n|,$$

which is equivalent to the true inequality

$$\sum_{n=1}^s \frac{c_n - \delta}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n - \delta}{\delta} |b_n| + \sum_{n=s+1}^{\infty} \frac{c_n - c_{s+1}}{\delta} |a_n| \geq 0. \quad (2.8)$$

If we take

$$f(z) = z + \frac{\delta}{c_{s+1}} z^{-s-1}, \quad (2.9)$$

with $z = re^{\frac{i\pi}{s+2}}$ and let $r \rightarrow 1^+$, we obtain

$$\frac{f(z)}{f_s(z)} = 1 + \frac{\delta z^{-s-2}}{c_{s+1}} \rightarrow 1 - \frac{\delta}{c_{s+1}},$$

which shows that the bound in (2.1) is best possible.

(ii) Similarly, if we put

$$\begin{aligned} g_2(z) &= \left(\frac{\delta + c_{s+1}}{\delta} \right) \left(\frac{f_s(z)}{f(z)} - \frac{c_{s+1}}{\delta + c_{s+1}} \right) \\ &= 1 - \frac{\left(\frac{\delta + c_{s+1}}{\delta} \right) \left(\sum_{n=s+1}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} \overline{b_n} z^{-n} \right)}{z + \sum_{n=1}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} \overline{b_n} z^{-n}}, \end{aligned}$$

and make use of (2.3), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{\left(\frac{c_{s+1} + \delta}{\delta} \right) \left(\sum_{n=s+1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right)}{2-2 \left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right) - \left(\frac{c_{s+1} - \delta}{\delta} \right) \left(\sum_{n=s+1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right)}. \quad (2.10)$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=1}^s |a_n| + \sum_{n=1}^{\infty} |b_n| + \left(\frac{c_{s+1}}{\delta} \right) \sum_{n=s+1}^{\infty} |a_n| \leq 1. \quad (2.11)$$

Since L.H.S. of (2.11) is bounded above by

$$\sum_{n=1}^{\infty} \frac{c_n}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n}{\delta} |b_n|,$$

the bound in (2.2) follows and is sharp with the extremal function $f(z)$ given by (2.4). The proof of Theorem 1 is now complete.

Employing the techniques used in Theorem 1, we can prove the following theorems.

Theorem 2. Let $r \in \mathbb{N}$ and let $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. Then

$$(i) \quad \operatorname{Re} \left\{ \frac{f(z)}{\tilde{f}_r(z)} \right\} > 1 - \frac{\delta}{d_{r+1}} \quad (z \in U), \quad (2.12)$$

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{\tilde{f}_r(z)}{f(z)} \right\} > \frac{d_{r+1}}{\delta + d_{r+1}} \quad (z \in U), \quad (2.13)$$

whenever

$$d_n \geq \begin{cases} \delta, & n = 2, 3, \dots, r, \\ d_{r+1}, & n = r + 1, r + 2, \dots \end{cases}$$

The estimates in (2.12) and (2.13) are sharp for the function given by

$$f(z) = z + \frac{\delta}{d_{r+1}} \bar{z}^{-r-1} \quad (z \in U^*). \quad (2.14)$$

Theorem 3. Let $s, r \in \mathbb{N}$ and let $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. Then

$$(i) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_{s,r}(z)} \right\} > 1 - \frac{\delta}{c_{s+1}} \quad (z \in U), \quad (2.15)$$

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{f_{s,r}(z)}{f(z)} \right\} > \frac{c_{s+1}}{\delta + c_{s+1}} \quad (z \in U), \quad (2.16)$$

whenever

$$\begin{aligned} c_n &\geq \begin{cases} \delta, & n = 2, 3, \dots, s, \\ c_{s+1}, & n = s + 1, s + 2, \dots, \end{cases} \\ d_n &\geq \begin{cases} \delta, & n = 2, 3, \dots, s, \\ c_{s+1}, & n = s + 1, s + 2, \dots \end{cases} \end{aligned} \quad (2.17)$$

Also,

$$(i) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_{s,r}(z)} \right\} > 1 - \frac{\delta}{d_{r+1}} \quad (z \in U), \quad (2.18)$$

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{f_{s,r}(z)}{f(z)} \right\} > \frac{d_{r+1}}{\delta + d_{r+1}} \quad (z \in U), \quad (2.19)$$

whenever

$$\begin{aligned} c_n &\geq \begin{cases} \delta, & n = 2, 3, \dots, r, \\ d_{r+1}, & n = r + 1, r + 2, \dots, \end{cases} \\ d_n &\geq \begin{cases} \delta, & n = 2, 3, \dots, r, \\ d_{r+1}, & n = r + 1, r + 2, \dots \end{cases} \end{aligned} \quad (2.20)$$

The estimates in (2.15), (2.16), (2.18) and (2.19) respectively, are sharp for the function given by (2.4) and (2.14), respectively.

Remark. By specializing the coefficients c_n, d_n and the parameters δ we obtain corresponding results for various subclasses mentioned in the introduction.

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