Journal of
Mathematics
and Applications

JMA No 37, pp 27-37 (2014)

On Generalised Quasi-ideals and Bi-ideals in Ternary Semigroups

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Abstract: In this paper, we introduce the notions of generalised quasi-ideals and generalised bi-ideals in a ternary semigroup. We also characterised these notions in terms of minimal quasi-ideals and minimal bi-ideals in a ternary semigroup.

AMS Subject Classification: *16Y30, 16Y60*
Keywords and Phrases: *Ternary Semigroup, Quasi-ideals, Bi-ideals, Minimal Quasi-ideals, Minimal Bi-ideals*

1 Introduction and Preliminaries

Good and Hughes[5] introduced the notion of bi-ideals and Steinfeld [2] introduced the notion of quasi-ideals in semigroups. In [1], Sioson studied the concept of quasi-ideals in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterised them by using the notion of quasi-ideals. In [7], Dixit and Dewan studied about the quasi-ideals and bi-ideals of ternary semigroups.

A nonempty set S with a ternary operation $S \times S \times S \longmapsto S$, written as $(x_1, x_2, x_3) \longmapsto [x_1x_2x_3]$ is called a ternary semigroup if it satisfies the following associative law: $[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$ for any $x_1, x_2, x_3, x_4, x_5 \in S$. In this paper, we denote $[x_1x_2x_3]$ by $x_1x_2x_3$.

A non-empty subset T of a ternary semigroup S is called a ternary subsemigroup if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$. A ternary subsemigroup I of a ternary semigroup S is called a left ideal of S if $SSI \subseteq I$, a lateral ideal if $SIS \subseteq I$, a right ideal of S if $ISS \subseteq I$, a two-sided ideal of S if I is both left and right ideal of S, and an ideal of S if I is a left, a right and a lateral ideal of S. An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$. Let S be a ternary semigroup. If there exists an element $0 \in S$ such that $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then "0" is called the zero element or simply the zero of the ternary semigroup S. In this case $S \cup \{0\}$ becomes a ternary semigroup with zero. For example, the set of all non-positive integers Z_0^{-1} forms a ternary semigroup with usual ternary multiplication and $'0'$ forms a ternary semigroup with zero element and also the zero element satisfy $(SS)^0S = S^0SS^0 = S(SS)^0 = S$. Throughout this paper S will always denote a ternary semigroup with zero. A ternary subsemigroup Q of a ternary semigroup S is called a quasi-ideal of S if $QSS \cap (SQS \cup SSQSS) \cap SSQ \subseteq Q$ and a ternary subsemigroup B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$. It is easy to see that every quasi-ideal in a ternary semigroup is a bi-ideal of S. An element a in a ternary semigroup S is called regular if there exists an element x in S such that $axa = a$. A ternary semigroup is called regular if all of its elements are regular. A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, lateral ideal M and left ideal L of S.

2 Generalised Quasi-ideals in Ternary Semigroup

In this section, we introduce the concept of generalised quasi-ideals in ternary semigroups and prove some results related to the same.

Definition 2.1. *A ternary subsemigroup Q of a ternary semigroup S is called a generalised quasi-ideal or $(m,(p,q),n)$-quasi-ideal of S if $Q(SS)^m \cap (S^pQS^q \cup S^pSQSS^q) \cap (SS)^nQ \subseteq Q$, where m, n, p, q are positive integers greater than 0 and $p + q = even$.*

Remark 2.1. *Every quasi-ideal of a ternary semigroup S is $(1,(1,1),1)$-quasi-ideal of S. But $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S need not be a quasi-ideal of S.*

Example 1. *Let $Z^- \setminus \{-1\}$ be the set of all negative integers excluding $\{0\}$. Then $Z^- \setminus \{-1\}$ is a ternary semigroup with usual ternary multiplication. Consider $Q = \{-3\} \cup \{k \in Z^- : k \leq -14\}$. Clearly Q is a non-empty ternary subsemigroup of S and also Q is $(2,(1,1),3)$-quasi-ideal of S. Now, $\{-12\} \in QSS \cap (SQS \cup SSQSS) \cap SSQ$. But $\{-12\} \notin Q$. Therefore $QSS \cap (SQS \cup SSQSS) \cap SSQ \not\subseteq Q$. Hence Q is not quasi-ideal of $Z^- \setminus \{-1\}$.*

Lemma 2.1. *Non-empty intersection of arbitrary collection of ternary subsemigroups of a ternary semigroup S is a ternary subsemigroup of S.*

Proof. Let T_i be a ternary subsemigroup of S for all $i \in I$ such that $\bigcap_{i\in I} T_i \neq \emptyset$. Let $t_1, t_2, t_3 \in \bigcap_{i\in I} T_i$. Then $t_1, t_2, t_3 \in T_i$ for all $i \in I$. Since T_i is a ternary subsemigroup of S for all $i \in I$, therefore $t_1t_2t_3 \in T_i$ for all $i \in I$. Therefore $t_1t_2t_3 \in \bigcap_{i\in I} T_i$. Hence $\bigcap_{i\in I} T_i$ is a ternary subsemigroup of S. □

Theorem 2.1. *Let S be a ternary semigroup and Q_i be an $(m,(p,q),n)$-quasi-ideal of S such that $\bigcap_{i\in I} Q_i \neq \emptyset$. Then $\bigcap_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Clearly $\bigcap_{i\in I} Q_i$ is a ternary subsemigroup of S (by Lemma 2.1).

Let $x \in \left[\bigcap_{i\in I} Q_i(SS)^m\right] \cap \left[S^p \bigcap_{i\in I} Q_iS^q \cup S^pS \bigcap_{i\in I} Q_iSS^q\right] \cap \left[(SS)^n \bigcap_{i\in I} Q_i\right]$. Then $x \in$

$\bigcap\limits_{i\in I} Q_i(SS)^m$, $x \in S^p \bigcap\limits_{i\in I} Q_iS^q \cup S^pS \bigcap\limits_{i\in I} Q_iSS^q$ and $x \in (SS)^n \bigcap\limits_{i\in I} Q_i$. This implies $x \in Q_i(SS)^m$, $x \in [S^pQ_iS^q \cup S^pSQ_iSS^q]$ and $x \in (SS)^nQ_i$ for all $i \in I$. Therefore $x \in [Q_i(SS)^m] \cap [S^pQ_iS^q \cup S^pSQ_iSS^q] \cap [(SS)^nQ_i] \subseteq Q_i$ for all $i \in I$, since Q_i is an $(m,(p,q),n)$-quasi-ideal of S. Thus $x \in Q_i$ for all $i \in I$. Therefore $x \in \bigcap\limits_{i\in I} Q_i$. Hence $\bigcap\limits_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Remark 2.2. *Let Z^- be the set of all negative integers under ternary multiplication and $Q_i = \{k \in Z^- : k \le -i\}$ for all $i \in I$. Then Q_i is an $(2,(1,1),3)$-quasi-ideal of Z^- for all $i \in I$. But $\bigcap\limits_{i\in I} Q_i = \emptyset$. So condition $\bigcap\limits_{i\in I} Q_i \neq \emptyset$ is necessary.*

Definition 2.2. *Let S be a ternary semigroup. Then a ternary subsemigroup*

(i) R of S is called an m-right ideal of S if $R(SS)^m \subseteq R$.
(ii) M of S is called an (p,q)-lateral ideal of S if $S^pMS^q \cup S^pSMSS^q \subseteq M$,
(iii) L of S is called an n-left ideal of S if $(SS)^nL \subseteq L$,

where m,n,p,q are positive integers and $p+q$ is an even positive integer.

Theorem 2.2. *Every m-right, (p,q)-lateral and n-left ideal of a ternary semigroup S is an $(m,(p,q),n)$-quasi-ideal of S. But converse need not be true.*

Proof. One way is straight forward. Conversely, let $S = M_2(Z_0^-)$ be the ternary semigroup of 2×2 square matrices over Z_0^-. Consider $Q = \left\{\begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix} : a \in Z_0^-\right\}$. Then Q is an $(2,(1,1),3)$-quasi-ideal of S. But it is not 2-right ideal, $(1,1)$-lateral ideal and 3-left ideal of S. □

Theorem 2.3. *Let S be a ternary semigroup. Then the following statements hold:*

(i) *Let R_i be an m-right ideal of S such that $\bigcap\limits_{i\in I} R_i \neq \emptyset$. Then $\bigcap\limits_{i\in I} R_i$ is an m-right ideal of S.*
(ii) *Let M_i be an (p,q)-lateral ideal of S such that $\bigcap\limits_{i\in I} M_i \neq \emptyset$. Then $\bigcap\limits_{i\in I} M_i$ is an (p,q)-lateral ideal of S.*
(iii) *Let L_i be an n-left ideal of S such that $\bigcap\limits_{i\in I} L_i \neq \emptyset$. Then $\bigcap\limits_{i\in I} L_i$ is an n-left ideal of S.*

Proof. Similar to the proof of Theorem 2.1 □

Theorem 2.4. *Let R be an m-right ideal, M be an (p,q)-lateral ideal and L be an n-left ideal of a ternary semigroup S. Then $R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Suppose $Q = R \cap M \cap L$. Since every m-right, (p,q)-lateral and n-left ideal of ternary semigroup S is an $(m,(p,q),n)$-quasi-ideal of S, therefore R, M and L are $(m,(p,q),n)$-quasi-ideals of S. Clearly, $R \cap M \cap L$ is non-empty. By Theorem 2.1, we have $Q = R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Lemma 2.2. *Let Q be an $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S. Then*

(i) $R = Q \cup Q(SS)^m$ *is an m-right ideal of S.*
(ii) $M = Q \cup (S^pQS^q \cup S^pSQSS^q)$ *is an (p,q)-lateral ideal of S.*
(iii) $L = Q \cup (SS)^nQ$ *is an n-left ideal of S.*

Proof. It is easy to show that R is ternary subsemigroup of S. Now to show that R is an m-right ideal of S.

$$\begin{aligned} R(SS)^m &= [(Q \cup Q(SS)^m](SS)^m \\ &= Q(SS)^m \cup Q(SS)^m(SS)^m \\ &= Q(SS)^m \cup Q(SSSS)^m \\ &\subseteq Q(SS)^m \cup Q(SS)^m \\ &= Q(SS)^m \subseteq R. \end{aligned}$$

Therefore R is an m-right ideal of S. Similarly, we can show that M is an (p,q)-lateral ideal of S and L is an n-left ideal of S. □

Theorem 2.5. *Every $(m,(p,q),n)$-quasi-ideal in a regular ternary semigroup S is the intersection of m-right, (p,q)-lateral and n-left ideal of S.*

Proof. Let S be regular ternary semigroup and Q be an $(m,(p,q),n)$-quasi-ideal of S. Then $R = Q \cup Q(SS)^m$, $M = Q \cup (S^pQS^q \cup S^pSQSS^q)$ and $L = Q \cup (SS)^nQ$ are m-right, (p,q)- lateral and n-left ideal of S respectively. Clearly $Q \subseteq R$, $Q \subseteq M$ and $Q \subseteq L$ implies $Q \subseteq R \cap M \cap L$. Since S is regular therefore $Q \subseteq Q(SS)^m$, $Q \subseteq S^pQS^q \cup S^pSQSS^q$ and $Q \subseteq (SS)^nQ$.
Thus $R = Q(SS)^m$, $M = S^pQS^q \cup S^pSQSS^q$ and $L = (SS)^nQ$. Now

$$R \cap M \cap L = Q(SS)^m \cap (S^pQS^q \cup S^pSQSS^q) \cap (SS)^nQ \subseteq Q$$

Hence, $Q = R \cap M \cap L$. □

3 Generalised Minimal Quasi-ideals

In this section, we study the concept of generalised minimal quasi-ideal or minimal $(m,(p,q),n)$-quasi-ideals of ternary semigroup S.

An $(m,(p,q),n)$-quasi-ideal Q of a ternary semigroup S is called minimal $(m,(p,q),n)$-quasi-ideal of S if Q does not properly contain any $(m,(p,q),n)$-quasi-ideal of S. Similarly, we can define minimal m-right ideals, minimal (p,q)-lateral ideals and minimal n-left ideals of a ternary semigroup.

Lemma 3.1. *Let S be a ternary semigroup and $a \in S$. Then the following statements hold:*

(i) $a(SS)^m$ *is an m-right ideal of S.*
(ii) $(S^paS^q \cup S^pSaSS^q)$ *is an (p,q)-lateral ideal of S.*

(iii) $(SS)^n a$ *is an* n*-left ideal of* S.

(iv) $a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a$ *is an* $(m,(p,q),n)$*-quasi-ideal of* S.

Proof. (i), (ii) and (iii) are obvious. (iv) follows from (i), (ii), (iii) and Theorem 2.4. □

Theorem 3.1. *Let* S *be a ternary semigroup and* Q *be an* $(m,(p,q),n)$*-quasi-ideal of* S*. Then* Q *is minimal iff* Q *is the intersection of some minimal* m*-right ideal* R*, minimal* (p,q)*-lateral ideal* M *and minimal* n*-left ideal* L *of* S.

Proof. Suppose Q is minimal $(m,(p,q),n)$-quasi-ideal of S. Let $a \in Q$. Then by above Lemma, we have $a(SS)^m$ is an m-right ideal, $(S^p aS^q \cup S^p SaSS^q)$ is an (p,q)-lateral ideal, $(SS)^n a$ is an n-left ideal and $a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a$ is an $(m,(p,q),n)$-quasi-ideal of S. Now,

$$\begin{aligned} & a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a \\ & \quad \subseteq Q(SS)^m \cap (S^p QS^q \cup S^p SQSS^q) \cap (SS)^n Q \\ & \quad \subseteq Q. \end{aligned}$$

Since Q is minimal therefore $a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a = Q$.
Now, to show that $a(SS)^m$ is minimal m-right ideal of S. Let R be an m-right ideal of S contained in $a(SS)^m$. Then

$$\begin{aligned} & R \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a \\ & \quad \subseteq a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a \\ & \quad = Q\,. \end{aligned}$$

Since $R \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a$ is an $(m,(p,q),n)$-quasi-ideal of S and Q is minimal, therefore $R \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a = Q$. This implies $Q \subseteq R$ and therefore

$$a(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$$

implies $R = a(SS)^m$. Thus m-right ideal $a(SS)^m$ is minimal. Similarly, we can prove that $(S^p aS^q \cup S^p SaSS^q)$ is minimal (p,q)-lateral ideal of S and $(SS)^n a$ is minimal n-left ideal of S.
Conversely, assume that $Q = R \cap M \cap L$ for some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L. So, $Q \subseteq R, Q \subseteq M$ and $Q \subseteq L$. Let Q' be an $(m,(p,q),n)$-quasi-ideal of S contained in Q. Then $Q'(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$. Similarly, $(S^p Q' S^q \cup S^p SQ' SS^q) \subseteq M$ and $(SS)^n Q' \subseteq (SS)^n Q \subseteq L$.
Now $Q'(SS)^m$ is an m-right ideal of S, as $Q'(SS)^m (SS)^m \subseteq Q'(SS)^m$. Similarly, $(S^p Q' S^q \cup S^p SQ' SS^q)$ is an (p,q)-lateral ideal of S and $(SS)^n Q'$ is an n-left ideal of S.
Since R, M and L are minimal m-right ideal, minimal (p,q)-lateral ideal and minimal n-left ideal of S respectively, therefore $Q'(SS)^m = R, S^p Q' S^q \cup S^p SQ' SS^q = M$ and $(SS)^n Q' = L$.
Thus $Q = R \cap M \cap L = Q'(SS)^m \cap (S^p Q' S^q \cup S^p SQ' SS^q) \cap (SS)^n Q' \subseteq Q'$. Hence $Q = Q'$. Thus Q is minimal $(m,(p,q),n)$-quasi-ideal of S. □

Note. *A ternary semigroup S need not contains a minimal $(m,(p,q),n)$-quasi-ideal of S.*
For example, let Z^- be the set of all negative integers. Then Z^- is a ternary semigroup with usual ternary multiplication. Let $Q = \{-2,-3,-4,\ldots\}$. Then Q is an $(2,(1,1),3)$-quasi-ideal of Z^-. Suppose Q is minimal $(2,(1,1),3)$-quasi-ideal of Z^-. Let $Q' = Q \setminus \{-2\}$. Then we can easily show that Q' is an $(2,(1,1),3)$-quasi-ideal of Z^-. But Q' is proper subset of Q. This is contradiction. Hence, Z^- does not contain a minimal $(m,(p,q),n)$-quasi-ideal.

Theorem 3.2. *Let S be a ternary semigroup. Then the following holds:*

(i) *An m-right ideal R is minimal iff $a(SS)^m = R$ for all $a \in R$.*
(ii) *An (p,q)-lateral ideal M is minimal iff $(S^p a S^q \cup S^p S a S S^q) = M$ for all $a \in M$.*
(iii) *An n-left ideal L is minimal iff $(SS)^n a = L$ for all $a \in L$.*
(iv) *An $(m,(p,q),n)$-quasi-ideal Q is minimal iff $a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a = Q$ for all $a \in Q$.*

Proof. (i) Suppose m-right ideal R is minimal. Let $a \in R$. Then $a(SS)^m \subseteq R(SS)^m \subseteq R$. By Lemma 3.1, we have $a(SS)^m$ is an m-right ideal of S. Since R is minimal m-right ideal of S therefore $a(SS)^m = R$.
Conversely, Suppose that $a(SS)^m = R$ for all $a \in R$. Let R' be an m-right ideal of S contained in R. Let $x \in R'$. Then $x \in R$. By assumption, we have $x(SS)^m = R$ for all $x \in R$. $R = x(SS)^m \subseteq R'(SS)^m \subseteq R'$. This implies $R \subseteq R'$. Thus, $R = R'$. Hence, R is minimal m-right ideal.
Similarly we can prove (ii), (iii) and (iv). □

4 Generalised Bi-ideals in Ternary Semigroup

In this section, we define generalised bi-ideals in a ternary semigroup and give their characterizations.

Definition 4.1. *A ternary subsemigroup B of a ternary semigroup S is called a generalised bi-ideal or $(m,(p,q),n)$ bi-ideal of S if $B(SS)^{m-1}S^p B S^q (SS)^{n-1} B \subseteq B$, where m,n,p,q are positive integers greater than zero and p and q are odd.*

Remark. *Every bi-ideal of a ternary semigroup S is $(1,(1,1),1)$-bi-ideal of S. But every $(m,(p,q),n)$-bi-ideal of a ternary semigroup S need not be a bi-ideal of S which is illustrated by the following example.*

Example 2. *Let $Z^- \setminus \{-1\}$ be the set of all negative integers excluding $\{0\}$. Then $Z^- \setminus \{-1\}$ is a ternary semigroup with usual ternary multiplication. Consider $B = \{-3,-27\} \cup \{k \in Z^- : k \leq -110\}$. Clearly B is a non-empty ternary subsemigroup of S and also B is $(3,(1,1),4)$-bi-ideal of S. Now $-108 \in BSBSB$. But $-108 \notin B$. Therefore $BSBSB \nsubseteq B$. Hence B is not a bi-ideal of $Z^- \setminus \{-1\}$.*

Theorem 4.1. *Let S be a ternary semigroup and B_i be an $(m,(p,q),n)$-bi-ideals of S such that $\bigcap_{i\in I} B_i \neq \emptyset$. Then $\bigcap_{i\in I} B_i$ is an $(m,(p,q),n)$ bi-ideal of S.*

Proof. It is straight forward. □

Remark. *Let Z^- be the set of all negative integers. Then Z^- is a ternary semigroup under usual ternary multiplication and $B_i = \{k \in Z^- : k \leq -i\}$ for all $i \in I$. Then B_i is an $(3,(1,1),4)$-bi-ideal of Z^- for all $i \in I$. But $\bigcap_{i\in I} B_i = \emptyset$. So condition $\bigcap_{i\in I} B_i \neq \emptyset$ is necessary.*

Theorem 4.2. *Every $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S is an $(m,(p,q),n)$-bi-ideal of S.*

Proof. Let Q be an $(m,(p,q),n)$-quasi-ideal of S. Then

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq Q(SS)^{m-1}S^pSS^q(SS)^{n-1}S \subseteq Q(SS)^m.$$

Similarly,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S(SS)^{m-1}(S^pQS^q)(SS)^{n-1}S \subseteq S^{p+1}QS^{q+1}.$$

Again $\{0\} \subseteq S^pQS^q$. So

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S^pQS^q \cup S^{p+1}QS^{q+1}.$$

Also,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S(SS)^{m-1}S^pSS^q(SS)^{n-1}Q \subseteq (SS)^nQ.$$

Consequently,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq Q(SS)^m \cap (S^pQS^q \cup S^{p+1}QS^{q+1}) \cap (SS)^nQ \subseteq Q.$$

Hence Q is an $(m,(p,q),n)$-bi-ideal of S. □

Remark. *Every $(m,(p,q),n)$-bi-ideal need not be an $(m,(p,q),n)$-quasi-ideal of S which is illustrated by the following example.*

Example 3. *Consider the ternary semigroup $S{=}Z^- \setminus \{-1\}$ with usual ternary multiplication and let $B = \{-3,-27\} \cup \{k \in Z^- : k \leq -194\}$. Clearly, B is non-empty ternary subsemigroup of S and also B is $(2,(1,1),3)$-bi-ideal of S. Now, $-192 \in B(SS)^2 \cap (SBS \cup SSBSS) \cap (SS)^3B$. But $-192 \notin B$. Therefore $B(SS)^2 \cap (SBS \cup SSBSS) \cap (SS)^3B \nsubseteq B$. Hence B is not $(2,(1,1),3)$-quasi-ideal of S.*

Theorem 4.3. *A ternary subsemigroup B of a regular ternary semigroup S is an $(m,(p,q),n)$-bi-ideal of S if and only if $B = BSB$.*

Proof. Suppose B is an $(m,(p,q),n)$-bi-ideal of a regular ternary semigroup S. Let $b \in B$. Then there exists $x \in S$ such that $b = bxb$. This implies that $b \in BSB$. Hence $B \subseteq BSB$. Now,

$$BSB \subseteq BSBSBSBSB \subseteq B(SS)(SBS)(SS)B \subseteq B.$$

Therefore $B = BSB$.

Conversely, if $B = BSB$, then

$$B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B(SS)^{m-1}S^pSS^q(SS)^{n-1}B \subseteq BSB = B.$$

Hence B is an $(m,(p,q),n)$-bi-ideal of S. □

Theorem 4.4. *Let S be a regular ternary semigroup. Then every $(m,(p,q),n)$-bi-ideal of S is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Let B be an $(m,(p,q),n)$-bi-ideal of S. Let $a \in B(SS)^m \cap (S^pBS^q \cup S^pSBSS^q) \cap (SS)^nB$. Then $a \in B(SS)^m, a \in (S^pBS^q \cup S^pSBSS^q)$ and $a \in (SS)^nB$. Thus $a = b(SS)^m = S^pb'S^q \cup S^pSb''SS^q = (SS)^nb'''$ for some $b, b', b'', b''' \in B$. Since S is regular, therefore for $a \in S$ there exists an element x in S such that $a = axa$. Then

$$\begin{aligned}
a = axa = axaxa \\
&= b(SS)^m x(S^pb'S^q \cup S^pSb''SS^q)x(SS)^nb''' \\
&\in B(SS)^m S(S^pBS^q \cup S^pSBSS^q)S(SS)^nB \\
&= [B(SS)^m SS^pBS^qS(SS)^nB] \cup [B(SS)^m SS^pSBSS^qS(SS)^nB] \\
&\subseteq B[(SS)^m SS^pSS^qS(SS)^n]B \cup B[(SS)^m SS^pSSSS^qS(SS)^n]B \\
&\subseteq BSB \cup BSB = B \cup B = B.
\end{aligned}$$

Thus $a \in B$. Therefore $B(SS)^m \cap (S^pBS^q \cup S^pSBSS^q) \cap (SS)^nB \subseteq B$. Hence B is an $(m,(p,q),n)$-quasi-ideal of S. □

It is easy to prove the following propositions:

Proposition 4.5. *The intersection of an $(m,(p,q),n)$-bi-ideal B of a ternary semigroup S with a ternary subsemigroup T of S is either empty or an $(m,(p,q),n)$-bi-ideal of T.*

Proposition 4.6. *Let B be an $(m,(p,q),n)$-bi-ideal of a ternary semigroup S and T_1, T_2 are two ternary subsemigroups of S. Then BT_1T_2, T_1BT_2 and T_1T_2B are $(m,(p,q),n)$-bi-ideals of S.*

Proposition 4.7. *Let B_1, B_2 and B_3 are three $(m,(p,q),n)$-bi-ideals of a ternary semigroup S. Then $B_1B_2B_3$ is an $(m,(p,q),n)$-bi-ideal of S.*

Proposition 4.8. *Let Q_1, Q_2 and Q_3 are three $(m,(p,q),n)$-quasi-ideals of a ternary semigroup S. Then $Q_1Q_2Q_3$ is an $(m,(p,q),n)$-bi-ideal of S.*

Proposition 4.9. *Let R be an m-right, M be an (p,q)-lateral and L be an n-left ideal of a ternary semigroup S. Then the ternary subsemigroup $B = RML$ of S is an $(m,(p,q),n)$-bi-ideal of S.*

Theorem 4.10. *Let S be a regular ternary semigroup. If B is an $(m,(p,q),n)$-bi-ideal of S, then $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B = B$.*

Proof. Let B be an $(m,(p,q),n)$-bi-ideal of S. Let $a \in B$. Then $a \in S$. Since S is regular, therefore there exists $x \in S$ such that $a = axa$. Now $a = axa = a(xa)(xax)(ax)a \in B(SS)(SBS)(SS)B$. Similarly, by property of regularity it is easy to show that $a \in B(SS)^{m-1}S^pBS^q(SS)^{n-1}B$. Thus, $B \subseteq B(SS)^{m-1}S^pBS^q(SS)^{n-1}B$. Since B is an $(m,(p,q),n)$-bi-ideal of S, therefore $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B$. Hence $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B = B$ □

Corollary 4.1. *Let S be a regular ternary semigroup. If Q is an $(m,(p,q),n)$-quasi-ideal of S, then $Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q = Q$.*

Proof. Since every $(m,(p,q),n)$-quasi-ideal of S is an $(m,(p,q),n)$-bi-ideal of S, therefore result follows directly. □

5 Generalised Minimal Bi-ideals

In this section, we introduce the concept of generalised minimal bi-ideal or minimal $(m,(p,q),n)$-bi-ideals in ternary semigroups.

Definition 5.1. *An $(m,(p,q),n)$-bi-ideal B of a ternary semigroup S is called minimal $(m,(p,q),n)$-bi-ideal of S if B does not properly contain any $(m,(p,q),n)$-bi-ideal of S.*

Lemma 5.1. *Let S be a ternary semigroup and $a \in S$. Then the following holds:*

(i) *$a(SS)^{m-1}$ is an m-right ideal of S.*
(ii) *S^paS^q is an (p,q)-lateral ideal of S.*
(iii) *$(SS)^{n-1}a$ is an n-left ideal of S.*
(iv) *$a(SS)^{m-1}S^paS^q\ (SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal.*

Proof. (i), (ii) and (iii) are obvious and (iv) follows from (i), (ii), (iii). □

Theorem 5.1. *Let S be a ternary semigroup and B be an $(m,(p,q),n)$-bi-ideal of S. Then B is minimal if and only if B is the product of some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L of S.*

Proof. Suppose B is minimal $(m,(p,q),n)$-bi-ideal of S. Let $a \in B$. Then by above Lemma, $a(SS)^{m-1}$ is an m-right ideal, S^paS^q is an (p,q)-lateral ideal, $(SS)^{n-1}a$ is an n-left ideal and $a(SS)^{m-1}S^paS^q(SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal of S. Now $a(SS)^{m-1}S^paS^q(SS)^{n-1}a \subseteq B(SS)^{m-1}S^pB\ S^q(SS)^{n-1}B \subseteq B$. Since B is minimal, therefore $a(SS)^{m-1}S^paS^q(SS)^{n-1}a = B$. Now to show that $a(SS)^{m-1}$ is minimal m-right ideal of S. Let R be an m-right ideal of S contained in $a(SS)^{m-1}$. Then $R(S^paS^q)(SS)^{n-1}a \subseteq a(SS)^{m-1}(S^paS^q)(SS)^{n-1}a = B$. Since $R\ S^paS^q(SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal of S and B is minimal, therefore $R(S^paS^q)(SS)^{n-1}a = B$. This implies $B \subseteq R$. Therefore $a(SS)^{m-1} \subseteq B(SS)^{m-1} \subseteq R(SS)^{m-1} \subseteq R$. Thus $a(SS)^{m-1}$ is minimal. Similarly we can prove that S^paS^q is minimal (p,q)-lateral ideal of S and $(SS)^{n-1}a$ is minimal n-left ideal of S.

Conversely, assume that $B = RML$ for some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L. So $B \subseteq R$, $B \subseteq M$ and $B \subseteq L$. Let B' be an $(m,(p,q),n)$-bi-ideal of S contained in B. Then $B'(SS)^{m-1} \subseteq B(SS)^{m-1} \subseteq R(SS)^{m-1} \subseteq R$. Similarly, $S^pB'S^q \subseteq S^pBS^q \subseteq S^pMS^q \subseteq M$ and $(SS)^{n-1}B' \subseteq (SS)^{n-1}B \subseteq (SS)^{n-1}L \subseteq L$. Now, $B'(SS)^{m-1}(SS)^m \subseteq B'(SS)^{m-1}$. So $B'(SS)^{m-1}$ is an m-right ideal of S. Similarly $S^pB'S^q$ is an (p,q)-lateral ideal and $(SS)^{n-1}B'$ is an n-left ideal of S. Since R, M and L are minimal m-right ideal, minimal (p,q)-lateral ideal and minimal n-left ideal of S respectively, therefore $B'(SS)^{m-1} = R$, $S^pB'S^q = M$ and $(SS)^{n-1}B' = L$. Thus $B = RML = B'(SS)^{m-1}S^pB'S^q(SS)^{n-1}B' \subseteq B'$. Hence $B = B'$. Consequently, B is minimal $(m,(p,q),n)$-bi-ideal of S. □

Definition 5.2. *Let S be a ternary semigroup. Then S is called a bi-simple ternary semigroup if S is the unique $(m,(p,q),n)$-bi-ideal of S.*

Theorem 5.2. *Let S be a ternary semigroup and B be an $(m,(p,q),n)$-bi-ideal of S. Then B is a minimal $(m,(p,q),n)$-bi-ideal of S if and if B is a bi-simple ternary semigroup.*

Proof. Suppose B is a minimal $(m,(p,q),n)$-bi-ideal of S. Let C be an $(m,(p,q),n)$-bi-ideal of B. Then $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C \subseteq B$. By Proposition 4.9, BCC is an $(m,(p,q),n)$-bi-ideal of S. Therefore

$(BCC)(SS)^{m-1}S^p(BCC)S^q(SS)^{n-1}BCC \subseteq BCC \subseteq BBB \subseteq B$. Since B is minimal, therefore $BCC = B$. It is easy to show that $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C$ is an $(m,(p,q),n)$-bi-ideal of S.

Since B is minimal, therefore $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C = B$. This implies $B = C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C$. Hence $C = B$. Consequently, B is a bi-simple ternary semigroup.

Conversely, suppose B is a bi-simple ternary semigroup. Let C be an $(m,(p,q),n)$-bi-ideal of S such that $C \subseteq B$. Then

$C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C(SS)^{m-1}S^pCS^q(SS)^{n-1}C \subseteq C$

which implies that C is an $(m,(p,q),n)$-bi-ideal of B. Since B is bi-simple ternary semigroup, therefore $C = B$. Hence B is minimal. □

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DOI: 10.7862/rf.2014.3

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Received 5.02.2013, Accepted 25.10.2013