Measure of Noncompactness and Neutral Functional Differential Equations with State-Dependent Delay

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ABSTRACT: Our aim in this work is to study the existence of solutions of first and second order for neutral functional differential equations with state-dependent delay. We use the Mönch's fixed point theorem for the existence of solutions and the concept of measures of noncompactness.

1. Introduction

In this work we prove the existence of solutions of first and second order for neutral functional differential equation with state-dependent delay. Our investigations will be situated on the Banach space of real valued functions which are defined, continuous and bounded on a real axis $\mathbb{R}$. More precisely, we will consider the following problem

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A[y(t) - g(t, y_{\rho(t,y_t)})] + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \ \ t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2}$$

where $f, g : J \times \mathcal{B} \to E$ are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t to some abstract phases $\mathcal{B}$, to be specified later.

Later, we consider the following problem

$$\frac{d}{dt}[y'(t) - g(t, y_{\rho(t,y_t)})] = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \qquad (3)$$

$$y(t) = \phi(t), \ t \in (-\infty, 0], \ y'(0) = \varphi, \qquad (4)$$

where $f, g : J \times \mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \in \mathbb{R}}$, on E, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. For the both problems, we will use Mönch's fixed theorem and the concept of measures of noncompactness combined with the Corduneanu's compactness criteria.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works [2, 11, 13, 15, 19, 20, 21].

The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [5, 10, 24, 32, 33, 37]. The literature relative second order differential system with state-dependent delay is very restrict, and related this matter we only cite [34] for ordinary differential system and [22] for abstract partial differential systems. Recently, in [1, 6, 8, 12] the authors provided some global existence and stability results for various classes of functional evolution equations with delay in Banach and Fréchet spaces.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and its equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [16], Travis and Webb [36].

Our purpose in this work is consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Webb in [35, 36]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [7, 28] to the context of partial second order differential equations, see [35] pp. 557 and the referred papers for details.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et al.* [3], Alváres [4], Banaś and Goebel [9], Guo *et al.* [17], Mönch [29], Mönch and Von Harten [30], and the references therein.

The literature on neutral functional evolution equations with delay on unbounded intervals is very limited. Some of them are stated in the Fréchet space setting, while the present ones are stated in the Banach setting. In particular our results extend those considered on bounded intervals by Hernandez and Mckibben [23]. Thus, the present paper complements that study.

2. Preliminaries

In this section we present briefly some notations and definition, and theorem which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [18] and follow the terminology used in [26]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$l = \sup\{L(t) : t \in J\},$$

and

$$m = \sup\{M(t) : t \in J\}.$$

Remark 2.1.

1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

Example 2.2. (The phase space $(\mathbf{C_r} \times \mathbf{L^p(g, E)})$)

Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with zero Lebesgue's measure. The space $C_r \times L^p(g, E)$ consists of all classes of functions $\varphi : (-\infty, 0] \to \mathbb{R}$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g, E)$ is defined by

$$\|\phi\|_{\mathcal{B}} := \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

Assume that $g(\cdot)$ verifies the condition $(g-5), (g-6)$ and $(g-7)$ in the nomenclature [26]. In this case, $\mathcal{B} = C_r \times L^p(g, E)$ verifies assumptions $(A_1), (A_2), (A_3)$ see ([26] Theorem 1.3.8) for details. Moreover, when $r = 0$ and $p = 2$ we have that

$$H = 1, \; M(t) = \gamma(-t)^{\frac{1}{2}}, \; L(t) = 1 + \left(\int_{-t}^{0} g(\theta)d\theta \right)^{\frac{1}{2}}, \; t \geq 0.$$

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

By $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

Finally, by $BC' := BC([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

Definition 2.3. A map $f : J \times \mathcal{B} \to E$ is said to be Carathéodory if

(i) $t \to f(t, y)$ is measurable for all $y \in \mathcal{B}$.

(ii) $y \to f(t, y)$ is continuous for almost each $t \in J$.

Now let us recall some fundamental facts of the Kuratowski measure of noncompactness.

Definition 2.4. Let E be a Banach space and Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \bigcup_{i+1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties

- (a) $\alpha(B) = 0 \Leftrightarrow \bar{B}$ is compact (B is relatively compact).

- (b) $\alpha(B) = \alpha(\bar{B})$.

- (c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

- (d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

- (e) $\alpha(cB) = |c|\alpha(B); \; c \in \mathbb{R}$.

- (f) $\alpha(convB) = \alpha(B)$.

Theorem 2.5. *(Mönch fixed point)[29]*
Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{conv}N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.6. *(Corduneanu)[14]*
Let $D \subset BC([0, +\infty), E)$. Then D is relatively compact if the following conditions hold:

(a) *D is bounded in BC.*

(b) *The function belonging to D is almost equicontinuous on $[0, +\infty)$, i.e., equicontinuous on every compact of $[0, +\infty)$.*

(c) *The set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0, +\infty)$.*

(d) *The function from D is equiconvergent, that is, given $\epsilon > 0$, responds $T(\epsilon) > 0$ such that $|u(t) - \lim_{t \to +\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.*

3. The first order problem

In this section we give our main existence result for problem (1)-(2). Before starting and proving this result, we give the definition of the mild solution.

Definition 3.1. We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (1)-(2) if $y(t) = \phi(t)$, $t \in (-\infty, 0]$ and the restriction of $y(\cdot)$ to the interval $[0, +\infty)$ is continuous and satisfies the following integral equation:

$$y(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{\rho(t, y_t)}) + \int_0^t T(t - s)f(s, y_{\rho(s, y_s)})ds, \ t \in J. \quad (5)$$

Set
$$\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B}, \ \rho(s, \phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\| \leq \mathcal{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [26].

Lemma 3.3. *([25]) If $y : (-\infty, +\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + l \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Let us introduce the following hypotheses

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a uniformly continuous semigroup $T(t)$, $t \in J$. Let $M' = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$|f(t, u)| \leq k(t)\|u\|_{\mathcal{B}}, \ t \in J, \ u \in \mathcal{B}$$

and

$$k^* := \sup_{t \in J} \int_0^t k(s)ds < \infty.$$

(H_4) For each bounded set $B \subset \mathcal{B}$, and each $t \in J$ we have

$$\alpha(f(t, B)) \leq k(t)\alpha(B).$$

(H_5) The function $g : J \times \mathcal{B} \to E$ is Carathéodory there exists a continuous function $k_g : J \to [0, +\infty)$ such that

$$|g(t, u)| \leq k_g(t)\|u\|_{\mathcal{B}}, \text{for each } u \in \mathcal{B}$$

and

$$k_g^* := \sup_{t \in J} \int_0^t k_g(s)ds < \infty.$$

(H_6) For each bounded set $B \subset \mathcal{B}$, and each $t \in J$ we have

$$\alpha(g(t, B)) \leq k_g(t)\alpha(B).$$

(H_7) For any bounded set $B \subset \mathcal{B}$, the function $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous on each compact interval of $[0, +\infty)$.

Theorem 3.4. *Assume that $(H_1) - (H_7)$ and (H_ϕ) hold. If*

$$l(M'k^* + k_g) < 1, \tag{6}$$

then the problem (1)-(2) has at least one mild solution on BC.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $N : BC \to BC$ defined by:

$$(Ny)(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0], \\[2mm] T(t)\left[\phi(0) - g(0, \phi(0))\right] \\[2mm] +g(t, y_{\rho(t, y_t)}) + \displaystyle\int_0^t T(t-s)\ f(s, y_{\rho(s, y_s)})\ ds; & \text{if } t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0]; \\[2mm] T(t)\ \phi(0); & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0; & \text{if } t \in (-\infty, 0]; \\[2mm] z(t); & \text{if } t \in J. \end{cases}$$

If y satisfies (5), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)}) - T(t)g(0, \phi(0))$$

$$+ \int_0^t T(t-s)f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})ds, t \in J.$$

Set

$$BC_0' = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC_0'} = \sup\{|z(t)| : t \in J\}, \ z \in BC_0'.$$

BC_0' is a Banach space with the norm $\|\cdot\|_{BC_0'}$. We define the operator $\mathcal{A} : BC_0' \to BC_0'$ by:

$$\mathcal{A}(z)(t) = g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)}) - T(t)g(0, \phi(0))$$

$$+ \int_0^t T(t-s)f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})ds, \ t \in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Mönch's fixed point theorem. The operator $\mathcal{A}$ maps BC_0' into BC_0', indeed the map $\mathcal{A}(z)$ is continuous

on $[0, +\infty)$ for any $z \in BC_0'$, and for each $t \in J$ we have

$$
\begin{aligned}
|\mathcal{A}(z)(t)| \;\leq\; & |g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)})| + M'|g(0, \phi(0))| \\
& + \; M' \int_0^t |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
\leq\; & M'(k_g \|\phi\|_{\mathcal{B}}) + k_g \|z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)}\|_{\mathcal{B}} \\
& + \; M' \int_0^t k(s) \|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}} ds \\
\leq\; & M'(k_g \|\phi\|_{\mathcal{B}}) + k_g(l|z(t)| + (m + \mathcal{L}^\phi + l M' H)\|\phi\|_{\mathcal{B}}) \\
& + \; M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + l M' H)\|\phi\|_{\mathcal{B}}) ds.
\end{aligned}
$$

Set

$$ C_1 := (m + \mathcal{L}^\phi + l M' H)\|\phi\|_{\mathcal{B}}. $$

$$ C_2 := M'(k_g \|\phi\|_{\mathcal{B}}) + k_g(m + \mathcal{L}^\phi + l M' H)\|\phi\|_{\mathcal{B}}. $$

Then, we have

$$
\begin{aligned}
|\mathcal{A}(z)(t)| \;\leq\; & C_2 + k_g l |z(t)| + M' C_1 \int_0^t k(s) ds + M' \int_0^t l|z(s)|k(s) ds \\
\leq\; & C_2 + k_g l \|z\|_{BC_0'} + M' C_1 k^* + M' l \|z\|_{BC_0'} k^*.
\end{aligned}
$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that

$$ r \geq \frac{C_2 + M' C_1 k^*}{1 - l(M' k^* + k_g)}, $$

and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$ |\mathcal{A}(z)(t)| \leq \; C_2 + k_g l r + M' C_1 k^* + M' k^* l r. $$

Thus

$$ \|\mathcal{A}(z)\|_{BC_0'} \leq r, $$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Mönch's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence

of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}, s \in J$.

If $s \in J$ is such that $\rho(s,z_s) > 0$, then we have,

$$
\begin{aligned}
\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\
&\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},
\end{aligned}
$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,z_s) > 0$. Similarly, is $\rho(s,z_s) < 0$, we get

$$
\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0
$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s,z_s) = 0$. Finally,

$$
\begin{aligned}
&\quad |\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \\
&\leq |g(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| \\
&\quad + M' \int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&\leq |g(t, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - g(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \\
&\quad + M' \int_0^t |f(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.
\end{aligned}
$$

Then by $(H_2), (H_5)$ we have

$$
f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \text{ as } n \to \infty,
$$

$$
g(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) \to g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}), \text{ as } n \to \infty,
$$

and by the Lebesgue dominated convergence theorem we get,

$$
\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \to 0, \text{ as } n \to \infty.
$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$. This is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for

$b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have:

$$|\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)|$$
$$\leq \quad |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})|$$
$$+ \quad \|T(\tau_2) - T(\tau_1)\|_{B(E)} |g(0, \phi(0))|$$
$$+ \quad \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds$$
$$+ \quad \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds$$
$$\leq \quad |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})|$$
$$+ \quad \|T(\tau_2) - T(\tau_1)\|_{B(E)} (k_g \|\phi\|_{\mathcal{B}})$$
$$+ \quad \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds$$
$$+ \quad \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds$$
$$\leq \quad |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})|$$
$$+ \quad \|T(\tau_2) - T(\tau_1)\|_{B(E)} (k_g \|\phi\|_{\mathcal{B}})$$
$$+ \quad C_1 \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s) ds$$
$$+ \quad rl \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s) ds$$
$$+ \quad C_1 \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s) ds$$
$$+ \quad rl \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s) ds.$$

When $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $T(t)$ is uniformly continuous operator (see [31]) and since (H_7), this proves the equicontinuity.

     Step 4: $\mathcal{A}(B_r)$ is equiconvergent.
Let $t \in [0, +\infty)$ and $z \in B_r$, we have,

$$
\begin{aligned}
|\mathcal{A}(z)(t)| &\leq |g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)})| + M'|g(0, \phi(0))| \\
&+ M' \int_0^t |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
&\leq C_2 + k_g lr + M' C_1 \int_0^t k(s) ds + M' rl \int_0^t k(s) ds.
\end{aligned}
$$

Set

$$C_3 = C_2 + k_g lr + M' C_1 k^* + M' lr k^*.$$

Then we have
$$\lim_{t \to +\infty} |\mathcal{A}(z)(t)| \leq C_3.$$
Hence,
$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as } t \to +\infty.$$

Now let V be a subset of B_r such that $V \subset \overline{conv}(\mathcal{A}(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $\mathbb{R}$.

$$
\begin{aligned}
V(t) &\leq \alpha(\mathcal{A}(V)(t) \cup \{0\})\alpha(\mathcal{A}(V)(t)) \\
&\leq k_g \alpha(V(t)) + M' \int_0^t k(s)\alpha(V(s))ds \\
&\leq k_g v(t) + M' \int_0^t k(s)v(s)ds \\
&\leq l(k_g + M'k^*)\|v\|_\infty.
\end{aligned}
$$

This means that
$$\|v\|_\infty (1 - l(k^* M' + k_g)) \leq 0.$$

By (6) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E. As a consequence of Steps 1-4, with Lemma 2.6, and from Mönch's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (1)-(2). $\qquad\square$

4. The Second order problem

In this section we are going to study existence of mild solution for problem (3)-(4). Before we mention a few results and notations respect of the cosine function theory which are needed to establish our results. Along of this section, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \geq 0}$ on Banach space $(E, |\cdot|)$. We denote by $(S(t))_{t \geq 0}$ the sine function associated with $(C(t))_{t \geq 0}$ which is defined by $S(t)y = \int_0^t C(s)yds$, for $y \in E$ and $t \geq 0$.

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|y\|_A = |y| + |Ay|$, $y \in D(A)$. Moreover, in this work, X is the space formed by the vector $y \in E$ for which $C(\cdot)y$ is of class C^1 on $\mathbb{R}$. It was proved by Kisyński [27] that X endowed with the norm
$$\|y\|_X = |y| + \sup_{0 \leq t \leq 1} |AS(t)y|, \ y \in X,$$
is a Banach space. The operator valued function
$$G(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}$$

is a strongly continuous group of bounded linear operators on the space $X \times E$ generated by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

defined on $D(A) \times X$. It follows this that $AS(t) : X \to E$ is a bounded linear operator and that $AS(T)y \to 0$, $t \longrightarrow 0$, for each $y \in X$. Furthermore, if $y : [0, +\infty) \to E$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)y(s)ds$ defined an X−valued continuous function. This is a consequence of the fact that:

$$\int_0^t G(t-s) \begin{pmatrix} 0 \\ y(s) \end{pmatrix} ds = \begin{pmatrix} \int_0^t S(t-s)y(s)ds \\ \int_0^t C(t-s)y(s)ds \end{pmatrix}$$

defines an $X \times E-$ valued continuous function. The existence of solutions for the second order abstract Cauchy problem.

$$\begin{cases} y''(t) = Ay(t) + h(t), & t \in J := [0, +\infty); \\ y(0) = y_0, & y'(0) = y_1, \end{cases} \qquad (7)$$

where $h : J \to E$ is an integrable function has been discussed in [35]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [36].

Definition 4.1. The function $y(\cdot)$ given by:

$$y(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)h(s)ds, t \in J,$$

is called mild solution of (7).

Remark 4.2. When $y_0 \in X, y(\cdot)$ is continuously differentiable and we have

$$y'(t) = AS(t)y_0 + C(t)y_1 + \int_0^t C(t-s)h(s)ds.$$

For additional details about cosine function theory, we refer the reader to [35, 36].

4.1. Existence of mild solutions

Now we give our main existence result for problem (3)-(4). Before starting and proving this result, we give the definition of a mild solution.

Definition 4.3. We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (3)-(4) if $y(t) = \phi(t)$, $t \in (-\infty, 0]$, $y'(0) = \varphi$ and

$$y(t) = C(t)\phi(0) + S(t)[\varphi - g(0, \phi)]$$

$$+ \int_0^t C(t-s)g(s, y_{\rho(s, y_s)})ds + \int_0^t S(t-s)f(s, y_{\rho(s, y_s)})ds, \ t \in J. \qquad (8)$$

Let us introduce the following hypothesis:

(H) $A : D(A) \subset E \to E$ is the infinitesimal generator of a uniformly continuous cosine function $(C(t))_{t \geq 0}$. Let

$$M_C = \sup\{\|C(t)\|_{B(E)} : t \geq 0\}, \quad M' = \sup\{\|S(t)\|_{B(E)} : t \geq 0\}.$$

Theorem 4.4. *Assume that $(H), (H_2) - (H_6), (H_\phi)$ hold. If*

$$l(k^* M' + M k_g^*) < 1, \tag{9}$$

then the problem (3)-(4) has at least one mild solution on BC.

Proof. We transform the problem (3)-(4) into a fixed point problem. Consider the operator: $N : BC \to BC$ define by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ C(t)\phi(0) + S(t)[\varphi - g(0, \phi)] \\ + \int_0^t C(t-s)\, g(s, y_{\rho(s,y_s)})\, ds + \int_0^t S(t-s)\, f(s, y_{\rho(s,y_s)})\, ds, & \text{if } t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0]; \\ C(t)\, \phi(0); & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, $y'(0) = \varphi = z'(0) = \varphi_1$, we denote by $\bar{z}$ the function

$$\bar{z}(t) = \begin{cases} 0; & \text{if } t \in (-\infty, 0]; \\ z(t); & \text{if } t \in J. \end{cases}$$

If y satisfies (8), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(.)$ satisfies

$$\begin{aligned} z(t) &= S(t)[\varphi_1 - g(0, \phi)] + \int_0^t C(t-s)\, g(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})ds \\ &\quad + \int_0^t S(t-s)\, f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})\, ds, \ t \in J. \end{aligned}$$

Set

$$BC_0' = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC_0'} = \sup\{|z(t)| : t \in J\}, \ z \in BC_0'.$$

BC_0' is a Banach space with the norm $\|.\|_{BC_0'}$. We define the operator $\mathcal{A} : BC_0' \to BC_0'$ by:

$$
\begin{aligned}
\mathcal{A}(z)(t) &= S(t)[\varphi_1 - g(0,\phi)] + \int_0^t C(t-s)\, g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds \\
&+ \int_0^t S(t-s)\, f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds,\ t \in J.
\end{aligned}
$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Mönch's fixed point theorem. The operator $\mathcal{A}$ maps BC_0' into BC_0', indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in BC_0'$, and for each $t \in J$ we have

$$
\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\
&+ M' \int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&+ M \int_0^t |g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\
&+ M' \int_0^t k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds \\
&+ M \int_0^t k_g(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\
&+ M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}})ds \\
&+ M \int_0^t k_g(s)(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}})ds.
\end{aligned}
$$

Let

$$
C = (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}.
$$

Then, we have:

$$
\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\
&+ M'C \int_0^t k(s)ds + M'l \int_0^t k(s)|z(s)|ds \\
&+ MC \int_0^t k_g(s)ds + Ml \int_0^t k_g(s)|z(s)|ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M'Ck^* + M'l\|z\|_{BC_0'}k^* \\
&+ MCk_g^* + Ml\|z\|_{BC_0'}k_g^*.
\end{aligned}
$$

Set

$$
C_1 = M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M'Ck^* + MCk_g^*.
$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that $r \geq \frac{C_1}{1-l(M'k^*+Mk_g^*)}$, and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|\mathcal{A}(z)(t)| \leq \ C_1 + M'lk^*r + Mlk_g^*r.$$

Thus,

$$\|\mathcal{A}(z)\|_{BC_0'} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Mönch's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence of the sequences $(z_{\rho(s,z_s^n)}^n)_{n \in \mathbb{N}}, s \in J$.
If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$
\begin{aligned}
\|z_{\rho(s,z_s^n)}^n - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s^n)}^n - z_{\rho(s,z_s^n)}\|_{\mathcal{B}} + \|z_{\rho(s,z_s^n)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\
&\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z_s^n)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},
\end{aligned}
$$

which proves that $z_{\rho(s,z_s^n)}^n \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get

$$\|z_{\rho(s,z_s^n)}^n - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi_{\rho(s,z_s^n)}^n - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z_{\rho(s,z_s^n)}^n \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z_{\rho(s,z_s)}^n \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$
\begin{aligned}
&|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \\
&\leq \ M' \int_0^t |f(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&+ \ M \int_0^t |g(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)}) - g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.
\end{aligned}
$$

Then by $(H_2), (H_5)$ we have

$$f(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \ \text{as } n \to \infty,$$

$$g(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)}) \to g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \ \text{as } n \to \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC_0'} \to 0, \ \text{as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$
\begin{aligned}
&|\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| \\
\leq\ & \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}[\|\varphi_1\| - g(0, \phi)] \\
+\ & \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})|ds \\
+\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|g(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|g(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})|ds \\
\leq\ & \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}[\|\varphi_1\| - g(0, \phi)] \\
+\ & C \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\ & lr \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\ & C \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}k(s)ds \\
+\ & lr \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}k(s)ds \\
+\ & C \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}k_g(s)ds \\
+\ & lr \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}k_g(s)ds \\
+\ & C \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}k_g(s)ds \\
+\ & lr \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}k_g(s)ds.
\end{aligned}
$$

When $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a uniformly continuous operator (see [35, 36]). This proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is equiconvergent.

Let $y \in B_r$, we have:

$$
\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M' \int_0^t |f(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)})| ds \\
&+ M \int_0^t |g(s, z_{\rho(s,z_s^n+x_s)}^n + x_{\rho(s,z_s^n+x_s)})| ds \\
&\leq C_1 + M'rl \int_0^t k(s)ds + Mrl \int_0^t k_g(s)ds.
\end{aligned}
$$

Then

$$
\lim_{t \to +\infty} |\mathcal{A}(z)(t)| \leq C_2,
$$

where

$$
C_2 \leq C_1 + rl(M'k^* + Mk_g^*).
$$

Hence,

$$
|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as} \ \ t \to +\infty.
$$

Now let V be a subset of B_r such that $V \subset \overline{conv}(\mathcal{A}(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $\mathbb{R}$.

$$
\begin{aligned}
V(t) &\leq \alpha(\mathcal{A}(V)(t) \cup \{0\}) \leq \alpha(\mathcal{A}(V)(t)) \\
&\leq M \int_0^t k_g(s)\alpha(V(s))ds + M' \int_0^t k(s)\alpha(V(s))ds \\
&\leq M \int_0^t k_g(s)v(s)ds + M' \int_0^t k(s)v(s)ds \\
&\leq l(Mk_g^* + M'k^*)\|v\|_\infty.
\end{aligned}
$$

This means that

$$
\|v\|_\infty(1 - l(k_g^*M + M'k^*)) \leq 0
$$

By (9) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E. From Mönch's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (3)-(4). $\qquad \square$

5. Examples

5.1. Example 1

Consider the following neutral functional partial differential equation:

$$
\frac{\partial}{\partial t}[z(t,x) - g(t, z(t - \sigma(t, z(t,0)), x))] = \frac{\partial^2}{\partial x^2}[z(t,x) - g(t, z(t - \sigma(t, z(t,0)), x))]
$$

$$+f(t, z(t - \sigma(t, z(t, 0)), x)), \ x \in [0, \pi], \ t \in [0, +\infty) \tag{10}$$

$$z(t, 0) = z(t, \pi) = 0, \ t \in [0, +\infty), \tag{11}$$

$$z(\theta, x) = z_0(\theta, x), \ t \in (-\infty, 0], \ x \in [0, \pi], \tag{12}$$

where f, g are given functions, and $\sigma : \mathbb{R} \to \mathbb{R}^+$. Take $E = L^2[0, \pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{are absolutely continuous}, \ \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n) \omega_n, \omega \in D(A),$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [31]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n) \omega_n, \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ the space of uniformly continuous and bounded functions from $\mathbb{R}^-$ into E and $\phi \in \mathcal{B}$, then (H_ϕ), where $\rho(t, \varphi) = t - \sigma(\varphi)$.

Hence, the problem (1)-(2) in an abstract formulation of the problem (10)-(12), and if the conditions $(H_1) - (H_7)$, (H_ϕ) are satisfied. Theorem 3.4 implies that the problem (10)-(12) has at least one mild solutions on BC.

5.2. Example 2

Take $E = L^2[0, \pi]$; $\mathcal{B} = C_0 \times L^2(h, E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous}, \ \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t \in \mathbb{R}}$ on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator S(t) is compact, for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathbb{R}.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi) = \tilde{y}(\xi + t),$$

where $\tilde{y}$ is the extension of y with period 2π. Then

$$C(t) = \frac{1}{2} (\Phi(t) + \Phi(-t)), \ A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0, \pi) : y(0) = x(\pi) = 0\}.$$

Consider the functional partial differential equation of second order:

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}z(t,x) \ + \ \int_{-\infty}^{0} b(s-t)z(s - \rho_1(t)\rho_2(|z(t)|), x)ds\right] = \frac{\partial^2}{\partial x^2}z(t,x)$$

$$+ \int_{-\infty}^{0} a(s-t)z(s - \rho_1(t)\rho_2(|z(t)|), x)ds,$$

$$x \in [0, \pi], \ t \in J := [0, +\infty), \quad (13)$$

$$z(t, 0) = z(t, \pi) = 0, \ t \in [0, +\infty), \quad (14)$$

$$z(t, x) = \phi(t, x), \quad \frac{\partial z(0, x)}{\partial t} = \varphi(x), \ t \in [-r, 0], \ x \in [0, \pi], \quad (15)$$

where $\phi \in \mathcal{B}, \ \rho_i : [0, \infty) \to [0, \infty), \ a, b : \mathbb{R} \to \mathbb{R}$ be continuous, and

$$L_f = \int_{-\infty}^{0} \frac{a^2(s)}{2h(s)}ds < \infty, \ L_g = \int_{-\infty}^{0} \frac{b^2(s)}{2h(s)}ds < \infty.$$

Under these conditions, we define the functions $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t, \psi)(x) = \int_{-\infty}^{0} a(s)\psi(s, x)ds,$$

$$g(t, \psi)(x) = \int_{-\infty}^{0} b(s)\psi(s, x)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2(|\psi(0)|),$$

we have

$$\|f(t, \cdot)\|_{\mathfrak{B}(\mathcal{B}, E)} \le L_f, \text{ and } \|g(t, \cdot)\|_{\mathfrak{B}(\mathcal{B}, E)} \le L_g.$$

Then the problem (3)-(4) in an abstract formulation of the problem (13)-(15). If conditions $(H_2) - (H_6), (H_\phi)$ are satisfied, Theorem 4.4 implies that the problem (13)-(15) has at least one mild solution on BC.

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