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Fourier, Laguerre, Laplace Transforms with Applications

Arman Aghili

Abstract: In this article, the author considered certain time fractional equations using joint integral transforms. Transform method is a powerful tool for solving singular integral equations, integral equation with retarded argument, evaluation of certain integrals and solution of partial fractional differential equations. The obtained results reveal that the transform method is very convenient and effective. Illustrative examples are also provided.

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1. Introduction and Preliminaries

This article is devoted to the study of the Laguerre transforms and its basic operational properties. The joint Laplace-Laguerre transform can be used effectively to solve the heat conduction problem in a semi-infinite medium with variable thermal conductivity. Another application of the Laguerre transform is to solve the problem of oscillations of a very heavy chain with variable tension. Apart from the ordinary or partial derivatives which occur in elementary calculus various other types of derivatives are known in the fractional calculus literature. Examples of this type are, the Caputo fractional derivative, the Riemann-Liouville fractional derivative. In the last three decades, considerable progress has been made in the area of fractional derivatives and, in general, in the area of fractional calculus. Partial fractional differential equations play an important role in science, engineering and social sciences. Nowadays it is impossible to describe a viscoelastic process without using a fractional derivative. Many phenomena in fluid mechanics, physics, biology, engineering and other areas of the sciences can be successfully modeled by the use of fractional derivatives. At this point, it should be emphasized that several definitions have been proposed the fractional derivatives, among those the Caputo and Riemann-Liouville is the most popular. Among scientists and engineers the Caputo fractional derivatives are more popular. Fractional differential equations arise in the unification of wave and diffusion phenomenon. The time fractional heat conduction equation, which is a mathematical model of a wide range of important physical phenomena, is obtained from the classical heat equation by replacing the first time derivative by a fractional derivative of order $0 < \alpha < 1$.

1.1. Definitions and Notations

Definition 1.1. Fourier transform of the function $\psi(x)$ is defined as follows

$$\mathcal{F}\{\psi(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx}\psi(x)dx := \Psi(w). \tag{1.1}$$

If $\mathcal{F}\{\psi(x)\} = \Psi(w)$, then $\mathcal{F}^{-1}\{\Psi(w)\}$ is given by

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw}\Psi(w)dw, \tag{1.2}$$

where $\psi(x)$, $\Psi(w)$ are elements of the $\mathcal{S}(R)$, space of rapidly decreasing functions or Schwartz class[13].

Note. The vector space $\mathcal{S}(R)$ of the rapidly decreasing functions is closed under linear combinations and differentiation. Any function belongs to $\mathcal{S}(R)$ is integrable. A typical function in this space is $\exp(-|x|^2)$.

Lemma 1.1. (Convolution Theorem for the Fourier transform)
Let us assume that $\mathcal{F}[\phi(x)] = \Phi(w)$ and $\mathcal{F}[\psi(x)] = \Psi(w)$, then the convolution of two functions $\phi(x)$ and $\psi(x)$ is defined by the expression

$$\phi(x) * \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(\xi)\psi(x-\xi)d\xi,$$

and the Fourier transform of the convolution is as follows

$$\mathcal{F}[\phi(x) * \psi(x); x \to w] = \Phi(w)\Psi(w).$$

Lemma 1.2. *We have the following identities for the Fourier transform*

$$1. \qquad \mathcal{F}[e^{-a^2x^2}; x \to w] = \frac{1}{a\sqrt{2}} e^{-\frac{w^2}{4a^2}}.$$

$$2. \qquad \mathcal{F}[|x|^{-\alpha}; x \to w] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\alpha)}{|w|^{1-\alpha}} \sin(\frac{\pi\alpha}{2}).$$

$$3. \qquad \mathcal{F}[sgn(x); x \to w] = \sqrt{\frac{2}{\pi}}\frac{i}{w}.$$

$$4. \qquad \mathcal{F}[xsgn(x); x \to w] = -\sqrt{\frac{2}{\pi}}\frac{1}{w^2}.$$

$$5. \qquad \mathcal{F}[e^{-a|x|}; x \to w] = \sqrt{\frac{2}{\pi}}\frac{a}{w^2 + a^2}.$$

Proof. See [4, 9, 13].

Lemma 1.3. *Let us assume that* $\mathcal{F}[\phi(x)] = \Phi(w)$ *then we have the following Fourier transform identities*

$$1. \qquad \mathcal{F}[\phi(x-\beta); x \to w] = e^{i\beta w}\Phi(w).$$

$$2. \qquad \mathcal{F}[\int_a^x \phi(\xi)d\xi; x \to w] = \frac{1}{iw}\Phi(w).$$

Proof. See [4, 9].

Lemma 1.4. *Let us show that*

$$\mathcal{F}^{-1}[\int_{-\infty}^{+\infty} J_\nu(a\xi)\frac{e^{-|x|\sqrt{w^2+\lambda^2}}}{2\sqrt{\xi^2+\lambda^2}}\xi^{\nu+1}d\xi] = \sqrt{\frac{2}{\pi}}(\sqrt{x^2+\lambda^2})^\nu K_{\frac{\nu}{2}}(a\sqrt{x^2+\lambda^2}).$$

Note. In the above relation $J_\nu(.)$ and $K_\nu(.)$ are the Bessel function of the first kind of order ν and the modified Bessel function of the second kind of order ν respectively.
Proof. By definition of the inverse Fourier transform, the left hand side of the above relation can be rewritten as follows

$$L.H.S = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}[\int_{-\infty}^{+\infty} J_\nu(a\xi)\frac{e^{-|w|\sqrt{w^2+\lambda^2}}}{2\sqrt{\xi^2+\lambda^2}}\xi^{\nu+1}d\xi]dw.$$

At this point, changing the order of integration yields

$$L.H.S = \int_{-\infty}^{+\infty} \frac{J_\nu(a\xi)}{2\sqrt{\xi^2+\lambda^2}}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}e^{-|w|\sqrt{w^2+\lambda^2}}dw]\xi^{\nu+1}d\xi,$$

in view of Lemma 1.2. the value of the inner integral is $\sqrt{\frac{2}{\pi}}\frac{\sqrt{\xi^2+\lambda^2}}{x^2+(\sqrt{\xi^2+\lambda^2})^2}$. After substitution and simplifying we arrive at

$$L.H.S = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\xi^{\nu+1}J_\nu(a\xi)}{\xi^2+(\sqrt{x^2+\lambda^2})^2}d\xi = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} \frac{\xi^{\nu+1}J_\nu(a\xi)}{\xi^2+(\sqrt{x^2+\lambda^2})^2}d\xi$$

$$= \sqrt{\frac{2}{\pi}}(\sqrt{x^2+\lambda^2})^{\nu} K_{\frac{\nu}{2}}(2\sqrt{x^2+\lambda^2}).$$

$\square$

The Fourier transform provides a useful technique for the solution of certain singular integral equations. Let us state and prove the following lemma.

Lemma 1.5. *Let us consider the following singular integral equation*

$$|x+\lambda|^{-\alpha} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{|\xi - x|^{\beta}} d\xi, \qquad 0 < \alpha < \beta < 1, \lambda > 0.$$

Then the above singular integral equation has the formal solution as follows

$$\phi(x) = \sqrt{\frac{\pi}{2}} \frac{\cos(\frac{\pi\beta}{2})\Gamma(\beta)}{\cos(\frac{\pi\alpha}{2})(\beta-\alpha)\Gamma(\alpha)} \frac{|x+\lambda|^{\alpha-\beta-1}}{\Gamma(\beta-\alpha)}.$$

Proof. By applying the Fourier transform to each term in the integral equation and using the convolution Theorem and Lemma 1.2. the singular integral equation is converted into the following equation

$$\sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\alpha)}{|w|^{1-\alpha}} e^{-i\lambda w} \sin(\frac{\pi\alpha}{2}) = \Phi(w) \sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\beta)}{|w|^{1-\beta}} \sin(\frac{\pi\beta}{2}),$$

the above equation has the following solution

$$\Phi(w) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)|w|^{\beta-\alpha}} e^{-i\lambda w} \frac{\sin(\frac{\pi\alpha}{2})}{\sin(\frac{\pi\beta}{2})}.$$

At this point by applying the inverse Fourier transform, we obtain the formal solution as below

$$\phi(x) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} \frac{\sin(\frac{\pi\alpha}{2})}{\sin(\frac{\pi\beta}{2})} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw} |w|^{\alpha-\beta} e^{-i\lambda w} dw,$$

from which we deduce that

$$\phi(x) = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)} \frac{\cos(\frac{\pi\beta}{2})}{(\beta-\alpha)\cos(\frac{\pi\alpha}{2})} |x+\lambda|^{\alpha-\beta-1}.$$

$\square$

Let us consider the special case $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, \lambda = 1$ we are led to the singular integral equation

$$|x+1|^{-\frac{1}{3}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{|\xi - x|^{\frac{1}{2}}} d\xi,$$

the solution of which is given by

$$\phi(x) = \frac{\pi\sqrt{6}}{\Gamma(\frac{1}{6})} |x+1|^{-\frac{7}{6}}.$$

Lemma 1.6. *Let us consider the following integral equation with retarded argument*

$$\phi(x-\beta) = f(x) + \lambda \int_a^x \phi(\xi)d\xi, \qquad \lambda > 0.$$

Then the above integral equation has the formal solution as follows

$$\phi(x) = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1} f^{(k)}(x-k\beta)}{\lambda^{k+1}}.$$

Proof. Taking the Fourier transform of the above integral equation term-wise and in view of Lemma1.3. We get

$$e^{i\beta w}\Phi(w) = F(w) + \frac{\lambda}{iw}\Phi(w),$$

from which we deduce that

$$\Phi(w) = \frac{F(w)}{e^{i\beta w} - \frac{\lambda}{iw}} = -\frac{1}{\lambda}\sum_{k=0}^{+\infty}(\frac{iwe^{iw\beta}}{\lambda})^k F(w) = -\sum_{k=0}^{+\infty}\frac{(-1)^k}{\lambda^{k+1}}(-iw)^k e^{ik\beta w}F(w).$$

By taking the inverse Fourier transform we obtain

$$\phi(x) = -\sum_{k=0}^{+\infty}\frac{(-1)^k}{\lambda^{k+1}}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(x-k\beta)w}(-iw)^k F(w)dw.]$$

At this point using the fact that $f^{(k)}(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-iw)^k e^{-i\eta w}F(w)dw$, we have

$$\phi(x) = \sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{\lambda^{k+1}} f^{(k)}(x-k\beta)$$

□

Definition 1.2. The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\phi(t)$ is defined as follows [12]

$$D^{c,\alpha}_{a,t}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{1.3}$$

Definition 1.3. Laplace transform of the function $\phi(t)$ is defined as follows

$$\mathcal{L}\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt := \Phi(s). \tag{1.4}$$

The sufficient conditions for the existence of the Laplace transform are that the function $\phi(t)$ be of exponential order and be sectionally continous on every closed interval

$0 \leq t \leq \lambda$ for every positive λ. The inverse Laplace transform of $\Phi(s)$ may be expressed explicitly as a contour integral by considering s as a complex variable. If $\mathcal{L}\{\phi(t)\} = \Phi(s)$, then $\mathcal{L}^{-1}\{\Phi(s)\}$ is given by

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Phi(s)ds, \tag{1.5}$$

where $\Phi(s)$ is analytic in the region $\mathrm{R}e(s) > c$.
Note. A theorem due to Lerch states that if two functions have the same Laplace transform they defer by a null function $\mathcal{N}(t)$, which has the property that, for every $\lambda > 0$,

$$\int_0^\lambda \mathcal{N}(t)dt = 0.$$

Definition 1.4. The Laplace transform of the Caputo fractional derivatives of order non-integer α. The most important use of the Caputo fractional derivative is treated in initial value problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this kind of derivative.

$$\mathcal{L}\{D_{0,t}^{c,\alpha} f(t)\} = sF(s) - f(0+), 0 < \alpha < 1. \tag{1.6}$$

and generally [12]

$$\mathcal{L}\{D_{0,t}^{c,\alpha} f(t)\} = s^{\alpha-1}F(s) - \textstyle\sum_{k=0}^{k=m-1-k} s^{\alpha-1-k} f^k(0+), m-1 < \alpha < m. \tag{1.7}$$

Example 1.1. Using convolution theorem for the Laplace transform to show that

$$\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi = (\frac{1}{\nu} + \frac{1}{\lambda})\frac{J_{\nu+\lambda}(\alpha t)}{t}, \quad \nu, \lambda > 0.$$

Where $J_\nu(.)$ stands for the Bessel function of the first kind of order ν.
Solution. From table of the Laplace transforms it is well known that [7]

$$\mathcal{L}[\frac{J_\nu(at)}{t}] = \frac{(\sqrt{s^2+a^2}-s)^\nu}{\nu a^\nu}.$$

In view of the convolution Theorem for the Laplace transform we have the following

$$\mathcal{L}[\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi] = \mathcal{L}[\frac{J_\nu(at)}{t}]\mathcal{L}[\frac{J_\lambda(at)}{t}] = \frac{(\sqrt{s^2+a^2}-s)^\nu}{\nu a^\nu}\frac{(\sqrt{s^2+a^2}-s)^\lambda}{\lambda a^\lambda},$$

after simplifying we arrive at

$$\mathcal{L}[\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi] = \frac{(\sqrt{s^2+a^2}-s)^{\nu+\lambda}}{\nu\lambda a^{\nu+\lambda}} = (\frac{1}{\nu}+\frac{1}{\lambda})\frac{(\sqrt{s^2+a^2}-s)^{\nu+\lambda}}{(\nu+\lambda)a^{\nu+\lambda}}.$$

At this stage taking the inverse Laplace transform of the above relation, we obtain

$$\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi = \mathcal{L}^{-1}[(\frac{1}{\nu}+\frac{1}{\lambda})\frac{(\sqrt{s^2+a^2}-s)^{\nu+\lambda}}{(\nu+\lambda)a^{\nu+\lambda}}] = (\frac{1}{\nu}+\frac{1}{\lambda})\frac{J_{\nu+\lambda}(at)}{t}.$$

Example 1.2. Using Bromwich complex inversion formula and residue theorem to show that

$$\begin{aligned}\phi(t) &= \mathcal{L}^{-1}[\frac{s^m K_1(\alpha\sqrt{s})}{\sqrt{s}-\lambda}] \\ &= 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty}(-r)^m e^{-tr}\frac{\lambda J_1(a\sqrt{r})+\sqrt{r}Y_1(a\sqrt{r})}{\lambda^2+r}dr.\end{aligned}$$

$$\lambda > 0, \quad m = 0, 1, 2, 3, ...$$

Where $K_\nu(.)$ stands for the modified Bessel function of the second kind of order ν or Macdonald's function [1].

Solution. The transform function $\Phi(s) = \frac{s^m K_1(\alpha\sqrt{s})}{\sqrt{s}-\lambda}$ has a simple pole at $s = \lambda^2$ and branch point at $s = 0$. Then the inverse Laplace transform is

$$\phi(t) = \lim_{s\to\lambda^2}[(s-\lambda^2)\frac{s^m K_1(\alpha\sqrt{s})e^{st}}{\sqrt{s}-\lambda}]$$

$$+\frac{1}{\pi}\int_0^{+\infty}e^{-tr}Im[\lim_{\theta\to-\pi}\Phi(re^{i\theta})]dr,$$

let us evaluate each term as follows

$$\phi(t) = \lim_{s\to\lambda^2}(\sqrt{s}-\lambda)(\sqrt{s}+\lambda).\frac{s^m K_1(a\sqrt{s})e^{st}}{\sqrt{s}-\lambda}$$

$$+\frac{1}{\pi}\int_0^{+\infty}e^{-tr}Im[\lim_{\theta\to-\pi}\frac{(re^{i\theta})^m K_1(\alpha\sqrt{re^{i\theta}})}{\sqrt{re^{i\theta}}-\lambda}]dr.$$

After evaluation of the limits and simplifying we get

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{\pi}\int_0^{+\infty}e^{-tr}\Im[\frac{(-r)^m K_1(-ia\sqrt{r})}{\lambda+i\sqrt{r}}]dr,$$

or

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{\pi}\int_0^{+\infty}e^{-tr}\Im[\frac{(-r)^m(\lambda-i\sqrt{r})K_1(-ia\sqrt{r})}{\lambda^2+r}]dr.$$

At this point let us recall the following well-known identity for the Bessel's function

$$K_\nu(z) = \frac{i\pi}{2}e^{\frac{\imath\pi\nu}{2}}[J_\nu(e^{\frac{i\pi}{2}}z)+iY_\nu(e^{\frac{i\pi}{2}}z)].$$

Thus, we get

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty}e^{-tr}\Im[\frac{(-r)^m(i\lambda+\sqrt{r})(J_1(a\sqrt{r})+iY_1(a\sqrt{r}))}{\lambda^2+r}]dr.$$

After simplification we obtain

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty}(-r)^m e^{-tr}\frac{\lambda J_1(a\sqrt{r})+\sqrt{r}Y_1(a\sqrt{r})}{\lambda^2+r}dr.$$

Definition 1.5. The generalized Laguerre polynomials $L_n^\alpha(x)$ satisfy the linear differential equation with non-constant coefficients

$$xy'' + (\alpha + 1 - x)y + ny = 0.$$

The generating function is

$$(1-t)^{-(\alpha+1)}\exp(-\frac{tx}{1-t}) = \sum_{n=0}^{+\infty} L_n^\alpha(x)t^n, \quad |t| < 1.$$

upon comparing coefficients of t^n in the two series expasions of the generating function, we obtain

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(-x)^k}{k!(n-k)!(1+\alpha)_k}.$$

In special case $\alpha = 0$, we have

$$L_n^0(x) = L_n(x) = \sum_{k=0}^{n} \frac{n!(-x)^k}{(k!)^2(n-k)!} = \sum_{k=0}^{k=n} C_k^n \frac{(-x)^k}{k!},$$

from which we deduce that

$$\mathcal{L}[L_n(x); x \to s] = \frac{1}{s}(1-\frac{1}{s})^n.$$

Note. The most important application of the Laguerre polynomials is in the quantum-mechanical analysis of the hydrogen atom.

Corollary 1.1. *The following identities hold true*

$$1. \qquad e^t J_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{L_n(x)t^n}{n!}.$$

$$2. \qquad e^{-t} I_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{(-1)^n L_n(x)t^n}{n!}.$$

Proof. Part(1). In view of the Lerch's theorem, by taking the Laplace transform of both sides with respect to $x, x > 0$ after some manipulations we get the same result. Part(2). In part (1), let us change t to $-t$ and using the fact that $J_0(it) = I_0(t)$ we get

$$e^{-t}J_0(2\sqrt{-tx}) = e^{-t}I_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{L_n(x)(-t)^n}{n!} = \sum_{n=0}^{+\infty}(-1)^n \frac{L_n(x)t^n}{n!}.$$

□

Definition 1.6. We define the associated Laguerre transform of the function $\phi(x)$ as follows

$$L_{n,\alpha}[\phi(x)] = \Phi_L(n,\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha} L_n^{\alpha}(x)\phi(x)dx,$$

and the inverse transform

$$L_{n,\alpha}^{-1}[\Phi_L(n,\alpha)] = \phi(x) = \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x)\Phi_L(n,\alpha).$$

In special case $\alpha = 0$, we define the Laguerre transform of the function $\phi(x)$ as follows

$$L_n[\phi(x)] = \Phi_L(n) = \int_0^{+\infty} e^{-x} L_n(x)\phi(x)dx,$$

and the inverse transform

$$L_n^{-1}[\Phi_L(n)] = \phi(x) = \sum_{n=0}^{+\infty} L_n(x)\Phi_L(n).$$

Remark 1.1. Let us consider the generalized Laguerre differential equation in self-adjoint form,

$$[(x^{\alpha+1}e^{-x})y']' + nx^{\alpha}e^{-x}y = 0,$$

we note that

$$L_{n,\alpha}[xy'' + (\alpha+1-x)y'] = -nL_{n,\alpha}y.$$

Thus, the Laguerre transform is suited for application to partial differential equations containing terms of the type

$$\mathcal{M}\psi = x\frac{\partial^2\psi}{\partial x^2} + (\alpha+1-x)\frac{\partial\psi}{\partial x}.$$

In special case $\alpha = 0$ we have

$$\mathcal{M}\psi = x\frac{\partial^2\psi}{\partial x^2} + (1-x)\frac{\partial\psi}{\partial x}.$$

Lemma 1.7. *The following identities hold true*

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x)\theta^n = \frac{1}{1-\theta}e^{-\frac{\theta(x+\xi)}{1-\theta}} I_0(\frac{2\sqrt{\theta x\xi}}{1-\theta}).$$

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x) = e^{\xi}\delta(x-\xi).$$

Proof. See [6].

Corollary 1.2. *In the first part of the above Lemma, if we set $\theta = e^{-t}$ we get the following result*

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x)e^{-nt} = \frac{1}{1-e^{-t}}e^{-\frac{e^{-t}(x+\xi)}{1-e^{-t}}} I_0(\frac{2\sqrt{e^{-t}x\xi}}{1-e^{-t}}).$$

2. Main Result (Exact solution to non-homogenous time fractional PDE via the Joint Laplace-Laguerre transforms)

The fractional derivatives are powerful technique for solving differential equations resulted from several physical modeling such as the fractional diffusion-wave equation, for more details see Mainardi [10, 11], Das [8]. However, some other researchers worked on the existence and uniqueness of solutions to some differential equations with fractional order (see Podlubny [12]).
In [2, 5] the author has used operational method to find analytical solutions of certain partial fractional differential equations. In this section, the author implemented the joint Laplace-Laguerre transform to construct exact solution for a variant of the time fractional heat conduction equation with non-constant coefficients and non-homogeneous initial condition.

Problem 1. Let us consider the following time fractional PDE with non-constant coefficints

$$D_t^{C,\alpha}u(x,t) = xu_{xx} + (\lambda+1-x)u_x, \quad 0<\alpha<1, \quad x,t>0.$$

$$u(x,0) = g(x), \quad |u(x,t)| < e^{kx}, k>1, x\to+\infty.$$

Solution. Let us define the joint Laplace-Laguerre transforms of the function $u(x,t)$ as follows

$$\mathcal{L}[L_{n,\lambda}u(x,t); x\to n]; t\to s] = U_{n,\lambda}(n,s) = \int_0^{+\infty} e^{-st}[\int_0^{+\infty} x^\lambda e^{-x} L_n^\lambda(x)u(x,t)dx]dt.$$

Note. It is worth mentioning that the joint Laplace-Laguerre transforms is very similar to the two dimensional Laplace transforms [6].
We find that the joint transforms applied to the problem leads to the transformed equation as below

$$s^\alpha U_{L,\lambda}(n,s) - s^{\alpha-1}G_{L,\lambda}(n) = -nU_{L,\lambda}(n,s),$$

where

$$U_{L,\lambda}(n,0) = G_{L,\lambda}(n) = \int_0^{+\infty} e^{-\xi}\xi^\lambda L_n^\lambda(\xi)g(\xi)d\xi.$$

From which we deduce that

$$U_{L,\lambda}(n,s) = \frac{s^{\alpha-1}G_{L,\lambda}(n)}{s^\alpha+n}.$$

At this stage, taking the inverse joint Laplace-Lageurre transforms we obtain

$$u(x,t) = \sum_{n=0}^{+\infty} \frac{n!L_n^\lambda(x)G_{L,\lambda}(n)}{\Gamma(\lambda+n+1)}[\mathcal{L}^{-1}(\frac{s^{\alpha-1}}{s^\alpha+n})].$$

Let us recall the Laplace transform of the pair of functions [12],

$$\int_0^{+\infty} e^{-st}E_\alpha(-nt^\alpha)dt = \frac{s^{\alpha-1}}{s^\alpha+n}, \quad E_\alpha(z) = \sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+1)}.$$

Note. $E_\alpha(z)$ stands for the Mittag-Leffler function. The Mittag-Leffler function is the basis function of the fractional calculus. Several modifications of the Mittag-Leffler functions are introduced for study of the fractional calculus [12].
Thus, we have

$$u(x,t) = \sum_{n=0}^{+\infty}\frac{n!L_n^\lambda(x)G_{L,\lambda}(n)}{\Gamma(\lambda+n+1)}E_\alpha(-nt^\alpha),$$

and so we finally deduce that

$$u(x,t) = \int_0^{+\infty} e^{-\xi}\xi^\lambda g(\xi)[\sum_{n=0}^{+\infty}\frac{n!L_n^\lambda(x)L_n^\lambda(\xi)}{\Gamma(\lambda+n+1)}E_\alpha(-nt^\alpha)]d\xi.$$

Let us study the following special cases
1. $\lambda=0,\ \alpha=1,\ g(x)=\delta(x-\beta)$, we get

$$u(x,t) = \int_0^{+\infty} e^{-\xi}g(\xi)[\sum_{n=0}^{+\infty}L_n(x)L_n(\xi)e^{-nt}]d\xi = e^{-\beta}[\sum_{n=0}^{+\infty}L_n(x)L_n(\beta)e^{-nt}].$$

After using the above Corollary 1.2. we get finally

$$u(x,t) = \frac{e^{-\beta}}{1-e^{-t}}e^{-\frac{e^{-t}(x+\beta)}{1-e^{-t}}}I_0(\frac{2\sqrt{e^{-t}x\beta}}{1-e^{-t}}).$$

Note. It is easy to verify that $u(x,0)=\delta(x-\beta)$, in view of Lemma 1.5. we have

$$u(x,0) = \int_0^{+\infty} e^{-\xi}\delta(\xi-\beta)[\sum_{n=0}^{+\infty}L_n(x)L_n(\beta)]d\xi = e^{-\beta}[\sum_{n=0}^{+\infty}L_n(x)L_n(\xi)]$$

$$= e^{-\beta}[e^\beta\delta(x-\beta)] = \delta(x-\beta).$$

2. For $\alpha=0.5,\ \lambda=0$, we have

$$u(x,t) = \sum_{n=0}^{+\infty}\frac{n!L_n(x)G_L(n)}{\Gamma(n+1)}[\mathcal{L}^{-1}(\frac{1}{\sqrt{s}(\sqrt{s}+n)})] = \sum_{n=0}^{+\infty}L_n(x)G_L(n)e^{n^2t}Erfc(n\sqrt{t}),$$

or

$$u(x,t) = \sum_{n=0}^{+\infty}L_n(x)G_L(n)e^{n^2t}Erfc(n\sqrt{t})$$
$$= \sum_{n=0}^{+\infty}L_n(x)e^{n^2t}Erfc(n\sqrt{t})[\int_0^{+\infty} e^{-\xi}g(\xi)L_n(\xi)d\xi].$$

Interchanging the order of the summation and integration, we arrive at

$$u(x,t) = \int_0^{+\infty} e^{-\xi} g(\xi)[\sum_{n=0}^{+\infty} L_n(x)L_n(\xi)e^{n^2 t} Erfc(n\sqrt{t})]d\xi,$$

in view of Lemma 1.5. we have

$$\begin{aligned} u(x,0) &= \int_0^{+\infty} e^{-\xi} g(\xi)[\sum_{n=0}^{+\infty} L_n(x)L_n(\xi) Erfc(n\sqrt{0})]d\xi \\ &= \int_0^{+\infty} e^{-\xi} g(\xi) e^{\xi} \delta(\xi - x) = g(x). \end{aligned}$$

3. Conclusion

The article is devoted to study and applications of the Fourier, Laplace and Laguerre transforms for solving certain singular integral equation, integral equation with retarded argument, and time fractional heat equation wth non-constant coefficients. The properties included in this article indicate the take-off points for advanced and modern developments in this field.

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Extremal Solutions to a Coupled System of Nonlinear Fractional Differential Equations with ψ–Caputo Fractional Derivatives

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Abstract: Using the well-known monotone iterative technique together with the method of upper and lower solutions, the authors investigate the existence of extremal solutions to a class of coupled systems of nonlinear fractional differential equations involving the ψ–Caputo derivative with initial conditions. As applications of this work, two illustrative examples are presented.

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Keywords and Phrases: ψ–Caputo fractional derivative; Coupled system; Extremal solutions; Monotone iterative technique; Upper and lower solutions.

1. Introduction

Recently, more and more attention has been given to the subject of fractional differential and integral equations due to their importance in applications in various branches of applied science and engineering; see, for example, [18, 26, 27, 31, 33, 34, 35]. For basic facts in the fractional calculus, we refer to the books [20, 28, 30, 45]. Almeida [8] introduced a new and general fractional derivative called the ψ–Caputo fractional derivative and extended the work of several researchers [19, 20, 25]. Additional details and properties of this fractional derivative can be found in [8, 9, 10, 11, 12]. Other important qualitative properties such as existence, uniqueness, and stability of solutions of various fractional differential problems can be found in the papers [4, 5, 6, 7, 21, 39, 40, 41]; see also [1, 2, 3, 45, 46].

The monotone iterative technique combined with the method of upper and lower solutions has been used by several authors to gain the existence and uniqueness of extremal solutions to nonlinear fractional differential equations (see, for instance, [13, 14, 15, 16, 22, 23, 24, 36, 37, 38, 42, 43, 44]).

Motivated by the above results, our goal is to extend the results in the recent paper by Derbazi et al. [15] by considering a more general system with the ψ–Caputo derivative. In [15], the authors considered the initial value problem

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}x(t) = f(t, x(t)), \quad t \in \mathrm{J} := [a, b], \\ x(a) = a^*, \end{cases} \tag{1}$$

where ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0, 1]$, $f \in C(\mathrm{J}\times\mathbb{R})$, and $a^* \in \mathbb{R}$. They were interested in the existence and uniqueness of extremal solutions to (1).

To the best of our knowledge, there are no known results on the existence of extremal solutions to systems of initial value problems for nonlinear fractional differential equations containing ψ–Caputo derivatives via the monotone iterative technique. As a result, we aim to fill this gap in the literature and contribute to enriching this area of research.

Here, we examine the existence of extremal solutions to the nonlinear coupled system

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}x(t) = f_1(t, x(t), y(t)), \quad x(a) = x_a, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi}y(t) = f_2(t, y(t), x(t)), \quad y(a) = y_a, \end{cases} \quad t \in \mathrm{J} := [a, b], \tag{2}$$

where ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0, 1]$, f_1, $f_2 \in C(\mathrm{J} \times \mathbb{R}^2, \mathbb{R})$, and x_a, $y_a \in \mathbb{R}$ with $x_a \leq y_a$.

The structure of this paper is as follows. In Section 2, we introduce definitions and preliminary results that will be needed to prove our main results. In Section 3, we apply the monotone iterative procedure and the method of upper and lower solutions to prove the existence of extremal solutions to the problem (2). In Section 4, we present two examples to illustrate the applicability of our results.

2. Preliminaries

In this section, we introduce some notations and definitions from the fractional calculus and present preliminary results needed in our proofs.

Let $J = [a, b]$, $0 < a < b < \infty$, be a finite interval and $\psi : [a, b] \to \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$.

We begin by defining ψ-Riemann-Liouville fractional integrals and derivatives. In what follows, $\Gamma(\cdot)$ is the (Euler's) Gamma function given by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}\mathrm{d}t, \quad \alpha > 0,$$

and $[\,\cdot\,]$ will denote the greatest integer function.

Definition 2.1. [8, 20] For $\alpha > 0$, the left-sided ψ–Riemann-Liouville fractional integral of order α of an integrable function $x\colon \mathrm{J} \longrightarrow \mathbb{R}$ with respect to the increasing differentiable function $\psi\colon \mathrm{J} \longrightarrow \mathbb{R}$ with $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$ is defined as

$$\mathbb{I}_{a^+}^{\alpha;\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) \mathrm{ds}.$$

Definition 2.2. [8] Let $n \in \mathbb{N}$ and let ψ, $x \in C^n(\mathrm{J}, \mathbb{R})$ be two functions such that ψ is increasing with $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$. The left-sided ψ–Riemann–Liouville fractional derivative of a function x of order α is defined by

$$\begin{aligned} \mathbb{D}_{a^+}^{\alpha;\psi} x(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \mathbb{I}_{a^+}^{n-\alpha;\psi} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} x(s) \mathrm{ds}, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3. [8] Let $n \in \mathbb{N}$ and let $\psi, x \in C^n(\mathrm{J}, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of x of order α is defined by

$$^c\mathbb{D}_{a^+}^{\alpha;\psi} x(t) = \mathbb{I}_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n x(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we will use the abbreviated symbol

$$x_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n x(t).$$

From the above definition, it is clear that

$$^c\mathbb{D}_{a^+}^{\alpha;\psi} x(t) = \begin{cases} \displaystyle\int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} x_\psi^{[n]}(s) \mathrm{ds}, & \text{if } \alpha \notin \mathbb{N}, \\ x_\psi^{[n]}(t), & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{3}$$

Lemma 2.1. *[10, 20] Let α, $\beta > 0$ and $u \in C(\mathrm{J}, \mathbb{R})$. Then for each $t \in \mathrm{J}$, we have:*

(1) $\mathbb{I}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\beta;\psi} u(t) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi} u(t)$,

(2) $^c\mathbb{D}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\alpha;\psi} x(t) = x(t)$,

(3) $\mathbb{I}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(a))^{\beta+\alpha-1}$,

(4) $^c\mathbb{D}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(a))^{\beta-\alpha-1}$,

(5) ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^k = 0,$ *for all* $k \in \{0,\dots,n-1\}, n \in \mathbb{N}$.

Next, we recall the composition rules for fractional ψ-integrals and ψ-derivatives.

Lemma 2.2. *[10] Let* $\alpha > 0$*. If* $x \in C^n(\mathrm{J},\mathbb{R})$ *and* $n-1<\alpha<n$*, then*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\,{}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{x_\psi^{[k]}(a)}{k!}\left[\psi(t)-\psi(a)\right]^k$$

for all $t \in \mathrm{J}$.

Definition 2.4 ([17])**.** The Mittag–Leffler functions of one and two parameters are defined respectively as

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in \mathbb{R} \text{ and } \alpha > 0,$$

and

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\ \beta > 0 \text{ and } z \in \mathbb{R}. \tag{4}$$

It is clear that $\mathbb{E}_{1,1}(z) = \mathbb{E}_1(z) = e^z$.

3. Main Results

In this section, we present the existence of extremal solutions to the system (2). The arguments are based on the monotone iterative technique combined with the method of upper and lower solutions. We begin by defining what we mean by a solution to (2).

Definition 3.1. By a solution of problem (2) we mean a pair of functions $(x,y) \in C(\mathrm{J},\mathbb{R}) \times C(\mathrm{J},\mathbb{R})$ that satisfies the system

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}x(t) = f_1(t,x(t),y(t)), \\ {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}y(t) = f_2(t,y(t),x(t)), \end{cases}$$

on J with the initial conditions

$$\begin{cases} x(a) = x_a, \\ y(a) = y_a. \end{cases}$$

To prove the existence of solutions to the problem (2), we need the following lemma.

Lemma 3.1. *[15] Let $\alpha \in (0,1]$ be fixed, $\lambda \in \mathbb{R}$, and $h \in C(\mathrm{J},\mathbb{R})$. Then, the linear problem*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} x(t) + \lambda x(t) = h(t), \ \ t \in \mathrm{J} := [a,b], \\ x(a) = x_a, \end{cases} \tag{5}$$

has a unique solution given by

$$x(t) = x_a \mathbb{E}_{\alpha,1}\big(-\lambda(\psi(t)-\psi(a))^{\alpha}\big) \tag{6}$$

$$+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-\lambda(\psi(t)-\psi(s))^{\alpha}\big)h(s)\mathrm{ds}, \tag{7}$$

where $\mathbb{E}_{\alpha,\beta}(\cdot)$ is the two-parametric Mittag–Leffer function defined in (4)*.*

Remark 3.1. Note that if $\lambda = 0$ in Lemma 3.1, we do not need the Mittag–Leffler function to compute the solution of the linear problem, and in fact, the unique explicit solution of (5) is given by

$$x(t) = x_a + \mathbb{I}_{a^+}^{\alpha;\psi} h(t).$$

As a consequence of Lemma 3.1, we have the following result that will be useful in proving our main theorem.

Lemma 3.2. *Let $\alpha \in (0,1]$ be fixed, λ, $\mu \in \mathbb{R}$, and h, $g \in C(\mathrm{J},\mathbb{R})$. Then the associated linear initial value problem*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} x(t) + \lambda x(t) + \mu y(t) = h(t), \quad x(a) = x_a, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi} y(t) + \lambda y(t) + \mu x(t) = g(t), \quad y(a) = y_a, \end{cases} \quad t \in \mathrm{J} := [a,b], \tag{8}$$

has a unique solution in $C(\mathrm{J},\mathbb{R}) \times C(\mathrm{J},\mathbb{R})$.

Proof. Let

$$x(t) = \frac{u(t)+v(t)}{2} \ \ \textit{and} \ \ y(t) = \frac{u(t)-v(t)}{2}.$$

Using (8), we have

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} u(t) + (\lambda+\mu)u(t) = (h+g)(t), \ \ t \in \mathrm{J} := [a,b], \\ u(a) = u_a = x_a + y_a, \end{cases} \tag{9}$$

and

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} v(t) + (\lambda-\mu)v(t) = (h-g)(t), \ \ t \in \mathrm{J} := [a,b], \\ v(a) = v_a = x_a - y_a. \end{cases} \tag{10}$$

By Lemma 3.1, we know that (9) and (10) have a unique solution $u \in C(\mathrm{J},\mathbb{R})$ and $v \in C(\mathrm{J},\mathbb{R})$, respectively, that can be expressed as

$$\begin{aligned} u(t) &= u_a \mathbb{E}_{\alpha}\big(-(\lambda+\mu)(\psi(t)-\psi(a))^{\alpha}\big) \\ &\quad + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-(\lambda+\mu)(\psi(t)-\psi(s))^{\alpha}\big)(h+g)(s)\mathrm{ds}, \\ v(t) &= v_a \mathbb{E}_{\alpha}\big(-(\lambda-\mu)(\psi(t)-\psi(a))^{\alpha}\big) \\ &\quad + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-(\lambda-\mu)(\psi(t)-\psi(s))^{\alpha}\big)(h-g)(s)\mathrm{ds}. \end{aligned}$$

Consequently, the linear system (8) has a unique solution (x, y). □

Next we present two important comparison results that will play important roles in the proof of our main result.

First, we state the following lemma that was proven in [15, Lemma 5].

Lemma 3.3 (Comparison result)**.** *Let $\alpha \in (0,1]$ be fixed and $\tau \in \mathbb{R}$. If $\theta \in C(\mathrm{J}, \mathbb{R})$ satisfies*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\theta(t) \geq -\tau\theta(t), \quad t \in [a,b], \\ \theta(a) \geq 0, \end{cases} \tag{11}$$

then $\theta(t) \geq 0$ for all $t \in \mathrm{J}$.

Based on the above lemma, we develop a new inequality involving the ψ–Caputo fractional derivative.

Lemma 3.4 (Comparison result)**.** *Let $\alpha \in (0,1]$ be fixed and $\lambda, \mu \in \mathbb{R}$ with $\mu \geq 0$. If $\rho, \nu \in C(\mathrm{J}, \mathbb{R})$ satisfy*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\rho(t) \geq -\lambda\rho(t) + \mu\nu(t), \quad \rho(a) \geq 0, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\nu(t) \geq -\lambda\nu(t) + \mu\rho(t), \quad \nu(a) \geq 0, \end{cases} \quad t \in \mathrm{J} := [a,b], \tag{12}$$

then $\rho(t) \geq 0$, $\nu(t) \geq 0$ for all $t \in \mathrm{J}$.

Proof. Let $\theta(t) = \rho(t) + \nu(t)$ for $t \in \mathrm{J}$. Then from (12),

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\theta(t) \geq -(\lambda - \mu)\theta(t), \\ \theta(a) \geq 0. \end{cases} \quad t \in \mathrm{J} := [a,b],$$

From Lemma 3.3, we obtain $\theta(t) \geq 0$, $t \in \mathrm{J}$, which implies that

$$\rho(t) + \nu(t) \geq 0 \;\text{ for }\; t \in \mathrm{J}. \tag{13}$$

Next, we will show that $\rho(t) \geq 0$ and $\nu(t) \geq 0$ for all $t \in \mathrm{J}$. In fact, from (12) and (13), we have

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\rho(t) + (\lambda + \mu)\rho(t) \geq 0, \quad \rho(a) \geq 0, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\nu(t) + (\lambda + \mu)\nu(t) \geq 0, \quad \nu(a) \geq 0. \end{cases} \quad t \in \mathrm{J} := [a,b], \tag{14}$$

It follows from inequalities (14) and Lemma 3.3 that

$$\rho(t) \geq 0 \quad \text{and} \quad \nu(t) \geq 0, \quad t \in \mathrm{J}.$$

□

Lemma 3.5. *Assume that $\{z_n(t)\}$ be a family of continuous functions on* J *satisfying*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} z_n(t) = f(t, z_n(t)), \\ z_n(a) = z_a, \end{cases} \quad t \in \mathrm{J} := [a, b], \tag{15}$$

for $n > 0$ and where $|f(t, z_n(t))| \leq M$ (M independent of n) for $t \in \mathrm{J}$. Then, the family $\{z_n(t)\}$ is equicontinuous on J.

Proof. According to Remark 3.1, the integral representation of (15) is given by

$$z_n(t) = z_a + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, z_n(s)) \mathrm{ds}. \tag{16}$$

For any t_1, $t_2 \in \mathrm{J}$ with $a < t_1 < t_2 < b$, from (16) we have

$$\begin{aligned} |z_n(t_2) - z_n(t_1)| \leq & \int_a^{t_1} \frac{\psi'(s) \left[(\psi(t_1) - \psi(s))^{\alpha-1} - (\psi(t_2) - \psi(s))^{\alpha-1}\right]}{\Gamma(\alpha)} |f(s, x_n(s)| \mathrm{ds} \\ & + \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_n(s))| \, \mathrm{ds} \\ \leq & \frac{M}{\Gamma(\alpha+1)} \left[(\psi(t_1) - \psi(a))^{\alpha} + 2(\psi(t_2) - \psi(t_1))^{\alpha} - (\psi(t_2) - \psi(a))^{\alpha}\right] \\ \leq & \frac{2M}{\Gamma(\alpha+1)} (\psi(t_2) - \psi(t_1))^{\alpha}. \end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero independently of $\{z_n\}$. Hence, the family $\{z_n(t)\}$ is equicontinuous on J. $\square$

Now, we are ready to establish our main theorem.

Theorem 3.1. *Let the functions f_1, $f_2 \in C(\mathrm{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, assume that:*

(H_1) *There exist x_0, $y_0 \in C(\mathrm{J}, \mathbb{R})$ with $x_0(t) \leq y_0(t)$ for $t \in \mathrm{J}$ such that*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} x_0(t) \leq f_1(t, x_0(t), y_0(t)), \\ x_0(a) \leq x_a, \end{cases} \quad t \in \mathrm{J} := [a, b] \tag{17}$$

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} y_0(t) \geq f_2(t, y_0(t), x_0(t)), \\ y_0(a) \geq y_a \end{cases} \quad t \in \mathrm{J} := [a, b]. \tag{18}$$

(H_2) *There exist constants $\lambda, \mu \in \mathbb{R}$ with $\mu \geq 0$ such that*

$$\begin{aligned} f_1(t, x, y) - f_1(t, \tilde{x}, \tilde{y}) &\geq -\lambda(x - \tilde{x}) - \mu(y - \tilde{y}), \\ f_2(t, y, x) - f_2(t, \tilde{y}, \tilde{x}) &\leq -\lambda(y - \tilde{y}) - \mu(x - \tilde{x}), \end{aligned}$$

where $x_0 \leq \tilde{x} \leq x \leq y_0$, $x_0 \leq y \leq \tilde{y} \leq y_0$, and

$$f_2(t, y, x) - f_1(t, x, y) \geq -\lambda(y - x) - \mu(x - y),$$

for $x_0 \leq x \leq y \leq y_0$.

Then the system (2) *has an extremal solution* $(x^*, y^*) \in [x_0, y_0] \times [x_0, y_0]$. *Moreover, there exist monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset [x_0, y_0]$ *that converge uniformly to* x^* *and* y^* *respectively, where* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *are defined by*

$$x_{n+1}(t) = \frac{u_{n+1}(t) + v_{n+1}(t)}{2}, \quad y_{n+1}(t) = \frac{u_{n+1}(t) - v_{n+1}(t)}{2}, \tag{19}$$

with

$$\begin{aligned} u_{n+1}(t) &= (x_a + y_a)\mathbb{E}_\alpha\big(-(\lambda+\mu)(\psi(t)-\psi(a))^\alpha\big) \\ &\quad + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-(\lambda+\mu)(\psi(t)-\psi(s))^\alpha\big) \\ &\quad \times \big(f_1(s, x_n(s), y_n) + f_2(s, y_n(s), x_n) + (\lambda+\mu)(x_n(s)+y_n(s))\big)\mathrm{ds}, \end{aligned}$$

$$\begin{aligned} v_{n+1}(t) &= (x_a - y_a)\mathbb{E}_\alpha\big(-(\lambda-\mu)(\psi(t)-\psi(a))^\alpha\big) \\ &\quad + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-(\lambda-\mu)(\psi(t)-\psi(s))^\alpha\big) \\ &\quad \times \big(f_1(s, x_n(s), y_n) - f_2(s, y_n(s), x_n) + (\lambda-\mu)(x_n(s)-y_n(s))\big)\mathrm{ds}, \end{aligned}$$

and

$$x_0(t) \le x_1(t) \le \cdots \le x_n(t) \le \cdots \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \quad t \in \mathrm{J}. \tag{20}$$

Proof. For any $x_0(t)$, $y_0(t) \in C(\mathrm{J}, \mathbb{R})$, we consider the linear system

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} x_{n+1}(t) = f_1(t, x_n(t), y_n(t)) - \lambda\big(x_{n+1}(t) - x_n(t)\big) - \mu\big(y_{n+1}(t) - y_n(t)\big), & t \in \mathrm{J}, \\ \quad x_{n+1}(a) = x_a, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi} y_{n+1}(t) = f_2(t, y_n(t), x_n(t)) - \lambda\big(y_{n+1}(t) - y_n(t)\big) - \mu\big(x_{n+1}(t) - x_n(t)\big), & t \in \mathrm{J}, \\ \quad y_{n+1}(a) = y_a. \end{cases} \tag{21}$$

By Lemma 3.1, we know that (21) has a unique solution in $C(\mathrm{J}, \mathbb{R}) \times C(\mathrm{J}, \mathbb{R})$ that are defined by (19). We complete the proof of the theorem through the following three steps.

Step 1: *The sequences* $\{x_n(t)\}$ *and* $\{y_n(t)\}$ *satisfy the relation*

$$x_n(t) \le x_{n+1}(t) \le y_{n+1}(t) \le y_n(t), \quad n = 0, 1, 2, \cdots, \ t \in \mathrm{J}. \tag{22}$$

First, we prove that

$$x_0(t) \le x_1(t) \le y_1(t) \le y_0(t), \quad t \in \mathrm{J}. \tag{23}$$

Set $\rho(t) = x_1(t) - x_0(t)$ and $\nu(t) = y_0(t) - y_1(t)$. From (21) and (H_1), we see that

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} \rho(t) \ge -\lambda\rho(t) + \mu\nu(t), & \rho(a) \ge 0, \\ {}^c\mathbb{D}_{a^+}^{\alpha;\psi} \nu(t) \ge -\lambda\nu(t) + \mu\rho(t), & \nu(a) \ge 0. \end{cases} \quad t \in \mathrm{J} := [a, b],$$

By Lemma 3.4, $\rho(t) \geq 0$ and $\nu(t) \geq 0$, for all $t \in \mathrm{J}$. That is, $x_0(t) \leq x_1(t)$. $y_1(t) \leq y_0(t)$, $t \in \mathrm{J}$.

Now, let $\theta(t) = y_1(t) - x_1(t)$. By (21) and (H_2), we have

$$\begin{aligned}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\theta(t) &= {}^c\mathbb{D}_{a^+}^{\alpha;\psi}y_1(t) - {}^c\mathbb{D}_{a^+}^{\alpha;\psi}x_1(t)\\
&= f_2(t, y_0(t), x_0(t)) - f_1(t, x_0(t), y_0(t)) + \lambda\big(y_0(t) - x_0(t)\big)\\
&\quad + \mu\big(x_0(t) - y_0(t)\big) - (\lambda - \mu)\big(y_1(t) - x_1(t)\big)\\
&\geq -\lambda(y_0(t) - x_0(t)) - \mu(x_0(t) - y_0(t)) + \lambda\big(y_0(t) - x_0(t)\big)\\
&\quad + \mu\big(x_0(t) - y_0(t)\big) - (\lambda - \mu)\theta(t)\\
&= -(\lambda - \mu)\theta(t).
\end{aligned}$$

Since, $\theta(a) = y_1(a) - x_1(a) = y_a - x_a \geq 0$. By Lemma 3.3, we obtain $x_1(t) \leq y_1(t)$, $t \in \mathrm{J}$.

Next, we show that $x_1(t)$ and $y_1(t)$ satisfy inequalities (17) and (18), respectively. Since x_0 and y_0 are respective solutions of (17) and (18), it follows that

$$\begin{cases}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}x_1(t) = f_1(t, x_0(t), y_0(t)) - \lambda\big(x_1(t) - x_0(t)\big) - \mu\big(y_1(t) - y_0(t)\big)\\
\qquad\qquad\quad \leq f_1(t, x_1(t), y_1(t))\\
\qquad x_1(a) \leq x_a
\end{cases}$$

and

$$\begin{cases}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}y_1(t) = f_2(t, y_0(t), x_0(t)) - \lambda\big(y_1(t) - y_0(t)\big) - \mu\big(x_1(t) - x_0(t)\big)\\
\qquad\qquad\quad \geq f_2(t, y_1(t), x_1(t))\\
\qquad y_1(a) \geq y_a.
\end{cases}$$

Therefore, $x_1(t)$ and $y_1(t)$ satisfy the inequalities (17) and (18), respectively.

By the above arguments and mathematical induction, the relation (22) holds.

Step 2: *The sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly to their limit functions x^* and y^*, respectively.* By (20), the sequences $\{x_n\}$ and $\{y_n\}$ are uniformly bounded on J. From Lemma 3.5, the sequences $\{x_n\}$ and $\{y_n\}$ are equicontinuous on J. Hence by the Ascoli-Arzelà Theorem, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ that converge uniformly to x^* and y^*, respectively, on J. This, together with the monotonicity of the sequences $\{x_n\}$ and $\{y_n\}$, implies

$$\lim_{n\to\infty} x_n(t) = x^*(t) \quad \text{and} \quad \lim_{n\to\infty} y_n(t) = y^*(t),$$

uniformly on $t \in \mathrm{J}$, and the limit functions x^*, y^* satisfy the problem (2).

Step 3: *System* (2) *has an extremal solution.* Assume that $(x(t), y(t)) \in [x_0(t), y_0(t)] \times [x_0(t), y_0(t)]$ be any solutions of system (2). That is,

$$\begin{cases}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}x(t) = f_1(t, x(t), y(t)), \quad x(a) = x_a,\\
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}y(t) = f_2(t, y(t), x(t)), \quad y(a) = y_a.
\end{cases} \quad t \in \mathrm{J} := [a, b],$$

We need to prove that $x^* \leq x$ and $y \leq y^*$; we do so by using induction. Clearly, $x_0(t) \leq x(t)$ and $y(t) \leq y_0(t)$. Assume that for some $n \in \mathbb{N}$,

$$x_n(t) \leq x(t) \quad \text{and} \quad y(t) \leq y_n(t), \quad t \in \mathrm{J}. \tag{24}$$

Let $\rho(t) = x(t) - x_{n+1}(t)$ and $\nu(t) = y_{n+1}(t) - y(t)$. From (21) and (H_2), we obtain

$$\begin{aligned}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\rho(t) &= {}^c\mathbb{D}_{a^+}^{\alpha;\psi}x(t) - {}^c\mathbb{D}_{a^+}^{\alpha;\psi}x_{n+1}(t) \\
&= f_1\big(t, x(t), y(t)\big) - f_1\big(t, x_n(t), y_n\big) + \lambda\big(x_{n+1}(t) - x_n(t)\big) \\
&+ \mu\big(y_{n+1}(t) - y_n(t)\big) \geq -\lambda(x(t) - x_n(t)) - \mu(y(t) - y_n(t)) \\
&+ \lambda\big(x_{n+1}(t) - x_n(t)\big) + \mu\big(y_{n+1}(t) - y_n(t)\big) \\
&= -\lambda\big(x(t) - x_{n+1}(t)\big) - \mu\big(y(t) - y_{n+1}(t)\big) = -\lambda\rho(t) + \mu\nu(t)
\end{aligned}$$

and

$$\rho(a) = x(a) - x_{n+1}(a) = x_a - x_a = 0.$$

Furthermore,

$$\begin{aligned}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\nu(t) &= {}^c\mathbb{D}_{a^+}^{\alpha;\psi}y_{n+1}(t) - {}^c\mathbb{D}_{a^+}^{\alpha;\psi}y(t) \\
&= f_2\big(t, y_n(t), x_n(t)\big) - f_2\big(t, y(t), x(t)\big) - \lambda\big(y_{n+1}(t) - y_n(t)\big) \\
&- \mu\big(x_{n+1}(t) - x_n(t)\big) \geq \lambda\big(y(t) - y_n(t)\big) + \mu\big(x(t) - x_n(t)\big) \\
&- \lambda\big(y_{n+1}(t) - y_n(t)\big) - \mu\big(x_{n+1}(t) - x_n(t)\big) \\
&= -\lambda\big(y_{n+1}(t) - y(t)\big) + \mu\big(x(t) - x_{n+1}(t)\big) = -\lambda\nu(t) + \mu\rho(t)
\end{aligned}$$

and

$$\nu(a) = y_{n+1}(a) - y(a) = y_a - y_a = 0.$$

Hence, the above argument yields

$$\begin{cases}
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\rho(t) \geq -\lambda\rho(t) + \mu\nu(t), & \rho(a) \geq 0, \\
{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\nu(t) \geq -\lambda\nu(t) + \mu\rho(t), & \nu(a) \geq 0.
\end{cases} \quad t \in \mathrm{J} := [a, b],$$

By Lemma 3.4, it follows that

$$x_{n+1}(t) \leq x(t) \quad \text{and} \quad y(t) \leq y_{n+1}(t), \ t \in \mathrm{J}.$$

Therefore, (24) holds on J for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ on both sides of (24), we get

$$x^* \leq x, \quad y \leq y^*.$$

Therefore (x^*, y^*) is an extremal solution of system (2) in $[x_0, y_0] \times [x_0, y_0]$. This completes the proof of the theorem □

4. Examples

In this section, we provide two examples to demonstrate the applicability of our results.

Example 4.1. Consider the fractional differential system

$$\begin{cases} {}^c\mathbb{D}_{0^+}^{0.5;\psi}x(t) = (t - x(t))^2 - 0.5ty(t), & x(0) = 0, \\ {}^c\mathbb{D}_{0^+}^{0.5;\psi}y(t) = (t - y(t))^2 - 0.5tx(t), & y(0) = 0, \end{cases} \quad t \in [0, 1], \tag{25}$$

where

$$\alpha = 0.5,\ a = 0,\ b = 1,\ \psi(t) = t,\ x_a = y_a = 0,$$

and

$$\begin{cases} f_1(t, x, y) = (t - x)^2 - 0.5ty, \\ f_2(t, y, x) = (t - y)^2 - 0.5tx. \end{cases} \quad t \in [0, 1],$$

Taking $x_0(t) = 0$ and $y_0(t) = t$, we obtain

$$\begin{cases} {}^c\mathbb{D}_{0^+}^{0.5;\psi}x_0(t) = 0 \leq 0.5t^2 = f_1(t, x_0(t), y_0(t)), & x_0(0) = 0, \\ {}^c\mathbb{D}_{0^+}^{0.5;\psi}y_0(t) = 2\sqrt{\frac{t}{\pi}} \geq 0 = f_2(t, y_0(t), x_0(t)), & y_0(0) = 0. \end{cases} \quad t \in [0, 1],$$

On the other hand, it is easily to verify that condition (H_2) holds for $\lambda = 2$ and $\mu = 0$. It follows from Theorem 3.1 that the nonlinear fractional differential system (25) has an extremal solution $(x^*, y^*) \in [0, t] \times [0, t]$. Furthermore, we have the iterative sequences

$$x_{n+1}(t) = \int_0^t (t - s)^{-0.5}\mathbb{E}_{0.5,0.5}\big(-2\sqrt{t - s}\big)\big((s - x_n(s))^2 - sy_n(s) + 2x_n(s)\big)\mathrm{ds},\ n \geq 0,$$

$$y_{n+1}(t) = \int_0^t (t - s)^{-0.5}\mathbb{E}_{0.5,0.5}\big(-2\sqrt{t - s}\big)\big((s - y_n(s))^2 - sx_n(s) + 2y_n(s)\big)\mathrm{ds},\ n \geq 0.$$

Example 4.2. Consider the coupled system

$$\begin{cases} {}^{CH}\mathbb{D}_{1^+}^{0.5;\psi}x(t) = 2\big(\ln^2(t) - x^2(t)\big) - \ln(t)y(t), & x(1) = 0, \\ {}^{CH}\mathbb{D}_{1^+}^{0.5;\psi}y(t) = 2\big(\ln^2(t) - y^2(t)\big) - \ln(t)x(t), & y(1) = 0, \end{cases} \quad t \in [1, e], \tag{26}$$

where

$$\alpha = 0.5,\ a = 1,\ b = e,\ \psi(t) = \ln(t),\ x_a = y_a = 0,$$

and

$$\begin{cases} f_1(t, x, y) = 2(\ln^2(t) - x^2) - \ln(t)y, \\ f_2(t, y, x) = 2(\ln^2(t) - y^2) - \ln(t)x. \end{cases} \quad t \in [1, e],$$

Taking $x_0(t) = 0$ and $y_0(t) = \ln(t)$, it is not difficult to show that (H1) holds. Also, it can easily be verified that condition (H_2) holds with $\lambda = 4$ and $\mu = 0$. Hence, the hypothesis of Theorem 3.1 are satisfied, and so the nonlinear fractional differential system (26) has an extremal solution $(x^*, y^*) \in [0, \ln(t)] \times [0, \ln(t)]$. Moreover, the monotone iterative sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ can be obtained by

$$\begin{aligned} x_{n+1}(t) = \int_1^t \left(\ln \frac{t}{s}\right)^{-0.5} \mathbb{E}_{0.5,0.5}\left(-4\sqrt{\ln \frac{t}{s}}\right)\big(2(\ln^2(s) - x_n^2(s)) \\ - \ln(s) y_n(s) + 4x_n(s)\big)\frac{\mathrm{ds}}{\mathrm{s}},\ n \geq 0, \\ y_{n+1}(t) = \int_1^t \left(\ln \frac{t}{s}\right)^{-0.5} \mathbb{E}_{0.5,0.5}\left(-4\sqrt{\ln \frac{t}{s}}\right)\big(2(\ln^2(s) - y_n^2(s)) \\ - \ln(s) x_n(s) + 4y_n(s)\big)\frac{\mathrm{ds}}{\mathrm{s}},\ n \geq 0. \end{aligned}$$

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Perturbation Theory, M-essential Spectra of 2×2 Operator Matrices and Application to Transport Operators

Aref Jeribi, Nedra Moalla and Sonia Yengui

Abstract: In this article we give some results on perturbation theory of 2×2 block operator matrices on the product of Banach spaces. Furthermore, we investigate their M-essential spectra. Finally, we apply the obtained results to determine the M-essential spectra of two group transport operators with general boundary conditions in the Banach space $L_p([-a,a] \times [-1,1]) \times L_p([-a,a] \times [-1,1])$, $p \geq 1$ and $a > 0$.

AMS Subject Classification: 39B42, 47A55, 47A53, 47A10.
Keywords and Phrases: Operator matrices; Essential spectra; Fredholm perturbation; Transport equation.

1. Introduction

Let X and Y be two Banach spaces. In this work we will discuss some results on perturbation theory of 2×2 operator matrices on $X \times Y$ and we will investigate their M-essential spectra. We consider operators in the following form

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are, in general, unbounded operators. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts between these spaces. Note that, in general L_0 is neither a closed nor a closable operator, even if its entries are closed. In [1], F. V. Atkinson,

H. Langer, R. Mennicken and A. A. Shkalikov give some sufficient conditions under which L_0 is closable and describe its closure, that we will denote by L.

In recent years, number of papers have been devoted to study the essential spectra of block operator matrices acting in a product of Banach spaces, (see [1], [2], [4], [5], [6], [14], [16], [23] and [26]). Most authors, there, have proposed methods for dealing with spectral theory for operators in the form $L_0 - \mu M$ where $M = I$. We note that the idea of studying the spectral characteristics of the 2×2 matrix operator goes back to the classics of the spectral theory for the differential operator. Hence several analysis focused on this issue may be found in the literature, see for example [10], [12], [13], [17], [18], [19], [20] and [25]. Recently, C. Tretter gives in [21], [22] and [23] an account research and presents a wide panorama of methods to investigate the spectral theory of block operator matrices. In the paper [8], M. Faierman, R. Mennicken and M. Möller propose a method for dealing with the spectral theory for pencils of the form $L_0 - \mu M$, where M is a bounded operator.

In this work, we generalize the results of [16] where M-essential spectra of some 2 $\times$ 2 operator matrices on $X \times X$ are discussed with $M = I$. For this, first we establish some results on perturbation theory of 2×2 operator matrices, essentially we prove the following result:

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2) \text{ if and only if } F_{ij} \in \mathcal{F}^b(X_j, X_i),\ \ i, j = 1, 2,$$

where $\mathcal{F}^b(X_j, X_i)$ designs the set of Fredholm perturbations (see Definition 2.2). Then we pursue the analysis started in [8] and we determine the M-essential spectra of a 2×2 matrix operator where M is a bounded operator defined on the product of two Banach spaces $X \times Y$.

We organize the paper in the following way: In Section 2, some preliminary abstract results about Fredholm operators are given. In Section 3, we establish some results on perturbation theory of 2×2 operator matrices. The Section 4 is devoted to the study of the M-essential spectra of a 2×2 matrix operator. Finally, in Section 5 we apply the obtained results to investigate the M-essential spectra of a two-group transport operator on L_p-spaces, $1 \leq p < \infty$.

2. Preliminary results

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ (resp. $\mathcal{C}(X, Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X, Y)$ the subspace of compact operators from X into Y. For $T \in \mathcal{C}(X, Y)$, we write $\mathcal{D}(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in Y.

Let S be a bounded operator from X to Y. For $T \in \mathcal{C}(X, Y)$, we define the

S-resolvent set of T by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ has a bounded inverse}\},$$

and the S-spectrum of T by:

$$\sigma_S(T) = \mathbb{C} \setminus \rho_S(T).$$

Now, we introduce the following important operator classes:

The set of upper semi-Fredholm operators is defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by:

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denote the set of Fredholm operators from X into Y and $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ the set of semi-Fredholm operators from X into Y. While the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T, for $T \in \Phi(X,Y)$. We say that the complex number λ is in $\Phi_{+T,S}$, $\Phi_{-T,S}$, $\Phi_{\pm T,S}$ or $\Phi_{T,S}$ if $\lambda S - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $C(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi_\pm(X)$, respectively.

In this paper we are concerned with the following S-essential spectra:

$$\begin{aligned}
&\sigma_{e_1,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_+(X,Y)\} := \mathbb{C} \setminus \Phi_{+T,S},\\
&\sigma_{e_2,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_-(X,Y)\} := \mathbb{C} \setminus \Phi_{-T,S},\\
&\sigma_{e_3,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_\pm(X,Y)\} := \mathbb{C} \setminus \Phi_{\pm T,S},\\
&\sigma_{e_4,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi(X,Y)\} := \mathbb{C} \setminus \Phi_{T,S},\\
&\sigma_{e_5,S}(T) := \mathbb{C} \setminus \rho_{5,S}(T),\\
&\sigma_{e_6,S}(T) := \mathbb{C} \setminus \rho_{6,S}(T),
\end{aligned}$$

where $\rho_{5,S}(T) := \{\lambda \in \Phi_{T,S} \text{ such that } i(\lambda S - T) = 0\}$ and $\rho_{6,S}(T)$ denote the set of those $\lambda \in \rho_{5,S}(T)$ such that all scalars near λ are in $\rho_S(T)$. They can be ordered as

$$\sigma_{e_3,S}(T) = \sigma_{e_1,S}(T) \cap \sigma_{e_2,S}(T) \subset \sigma_{e_4,S}(T) \subset \sigma_{e_5,S}(T) \subset \sigma_{e_6,S}(T).$$

Note that if $S = I$, we recover the usual definition of the essential spectra of a closed densely defined linear operator (see [16]).

Let us, now, introduce some notation and then continue with some lemmas and propositions.

Proposition 2.1. *[15] Let $T \in \mathcal{C}(X,Y)$ and consider S a nonzero bounded linear operator from X into Y. Then we have the following results:*
(i) $\Phi_{T,S}$ is open.
(ii) $i(\lambda S - T)$ is constant on any component of $\Phi_{T,S}$.
(iii) $\alpha(\lambda S - T)$ and $\beta(\lambda S - T)$ are constant on any component of $\Phi_{T,S}$ except on a discrete set of points on which they have larger values.

Definition 2.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. F is called strictly singular, if for every infinite-dimensional closed subspace $\mathcal{M}$ of X, the restriction of F to $\mathcal{M}$ is not bijective.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X to Y.

Definition 2.2. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.
(i) The operator F is called Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$.
(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp. $U+F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp. $U \in \Phi_-(X,Y)$).

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbations and by $\mathcal{F}_+(X,Y)$ (resp. $\mathcal{F}_-(X,Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations.

Remark 2.1. Let $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$ respectively. If in Definition 2.2 we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$.

The sets of Fredholm perturbations and semi-Fredholm perturbations were introduced in [9]. In particular it is shown that $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X) := \mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X) := \mathcal{F}^b_-(X,Y)$ are closed two-sided ideals of $\mathcal{L}(X)$.

In general, we have the following inclusions:

$$\mathcal{K}(X,Y) \subset \mathcal{S}(X,Y) \subset \mathcal{F}^b(X,Y).$$

Note that, the set $\mathcal{F}^b(X,Y)$ can strictly contains $\mathcal{S}(X,Y)$. Indeed, in [27], the author gives some geometric conditions on the Banach spaces for which the equality $\mathcal{S}(X,Y) = \mathcal{F}^b(X,Y)$ does not hold.
Recall the following result established in [3].

Lemma 2.1. *[3] Let X and Y be two Banach spaces, then*
$$\mathcal{F}(X,Y) = \mathcal{F}^b(X,Y).$$

Proposition 2.2. *[15] Let T_1, T_2 are two closed densely defined linear operators on X and S an invertible operator on X.*

(i) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b(X)$, then
$$\sigma_{e_i,S}(T_1) = \sigma_{e_i,S}(T_2),\ \ i = 4, 5.$$

(ii) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b_+(X)$, then
$$\sigma_{e_1,S}(T_1) = \sigma_{e_1,S}(T_2).$$

(iii) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b_-(X)$, then
$$\sigma_{e_2,S}(T_1) = \sigma_{e_2,S}(T_2).$$

(iv) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$*, the operator* $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}_+^b(X) \cap \mathcal{F}_-^b(X)$*, then*

$$\sigma_{e_3,S}(T_1) = \sigma_{e_3,S}(T_2).$$

Definition 2.3. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to have a left Fredholm inverse if there exists an operator $T_l \in \mathcal{L}(Y,X)$ such that $T_lT - I \in \mathcal{K}(X)$. Similarly, T is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y,X)$ such that $TT_r - I \in \mathcal{K}(Y)$. The operators T_l and T_r are called, respectively, left and right Fredholm inverse of T.

We will denote by $\Phi_l^b(X,Y)$ (resp. $\Phi_r^b(X,Y)$) the set of bounded operators which have left Fredholm inverse (resp. right Fredholm inverse).
It follows from [17, Theorems 14. and 15. p. 160] that

$$\Phi_l^b(X,Y) = \{T \in \Phi_+^b(X,Y) \text{ such that } R(T) \text{ is complemented}\}$$

and

$$\Phi_r^b(X,Y) = \{T \in \Phi_-^b(X,Y) \text{ such that } \ker(T) \text{ is complemented}\},$$

where a subspace $N \subset X$ is said to be complemented if there exists a closed subspace $M \subset X$ such that $N \oplus M = X$.
Note that we have the following inclusions:

$$\Phi^b(X,Y) \subset \Phi_l^b(X,Y) \subset \Phi_+^b(X,Y)$$

and

$$\Phi^b(X,Y) \subset \Phi_r^b(X,Y) \subset \Phi_-^b(X,Y).$$

Definition 2.4. Let X and Y be two Banach spaces. We denote by

$$\mathcal{F}_l^b(X,Y) = \{F \in \mathcal{L}(X,Y) \text{ such that } T + F \in \Phi_l^b(X,Y) \text{ whenever } T \in \Phi_l^b(X,Y)\}$$

and

$$\mathcal{F}_r^b(X,Y) = \{F \in \mathcal{L}(X,Y) \text{ such that } T + F \in \Phi_r^b(X,Y) \text{ whenever } T \in \Phi_r^b(X,Y)\}.$$

The set $\mathcal{F}_l^b(X,X)$ (resp. $\mathcal{F}_r^b(X,X)$) will be denoted by $\mathcal{F}_l^b(X)$ (resp. $\mathcal{F}_r^b(X)$).

Proposition 2.3. *Let* X*,* Y *and* Z *be three Banach spaces.*
(i) If $A \in \Phi^b(Y,Z)$ *and* $T \in \Phi_l^b(X,Y)$ *(resp.* $T \in \Phi_r^b(X,Y)$*), then* $AT \in \Phi_l^b(X,Z)$ *(resp.* $AT \in \Phi_r^b(X,Z)$*).*

(ii) If $A \in \Phi^b(X,Y)$ *and* $T \in \Phi_l^b(Y,Z)$ *(resp.* $T \in \Phi_r^b(Y,Z)$*), then* $TA \in \Phi_l^b(X,Z)$ *(resp.* $TA \in \Phi_r^b(X,Z)$*).*

Proof. (i) Let $A \in \Phi^b(Y,Z)$, then, by [24, Theorem 5.4.] there exist $A_0 \in \mathcal{L}(Z,Y)$ and $K_1 \in \mathcal{K}(Y)$ (resp. $K_2 \in \mathcal{K}(Z)$) such that $A_0A = I_Y - K_1$ (resp. $AA_0 = I_Z - K_2$). On the other hand, there exist $T_l \in \mathcal{L}(Y,X)$ (resp. $T_r \in \mathcal{L}(Y,X)$) and $K_3 \in \mathcal{K}(X)$ (resp. $K_4 \in \mathcal{K}(Y)$) such that $T_lT = I_X - K_3$ (resp. $TT_r = I_Y - K_4$) since $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$). So, $T_lA_0AT = I_X - K_3 - T_lK_1T$ (resp. $ATT_rA_0 = I_Z - K_4 - AK_2A_0$), which imply that $AT \in \Phi_l^b(X,Z)$ (resp. $AT \in \Phi_r^b(X,Z)$).
(ii) The proof is analogous to the previous one. □

Proposition 2.4. *Let X, Y and Z be three Banach spaces.*
(i) *If the set $\Phi^b(Y,Z)$ is not empty, then*

$$F_1 \in \mathcal{F}_l^b(X,Y) \text{ and } A \in \Phi^b(Y,Z), \text{ imply } AF_1 \in \mathcal{F}_l^b(X,Z),$$

$$F_1 \in \mathcal{F}_r^b(X,Y) \text{ and } A \in \Phi^b(Y,Z), \text{ imply } AF_1 \in \mathcal{F}_r^b(X,Z).$$

(ii) *If the set $\Phi^b(X,Y)$ is not empty, then*

$$F_2 \in \mathcal{F}_l^b(Y,Z) \text{ and } A \in \Phi^b(X,Y), \text{ imply } F_2A \in \mathcal{F}_l^b(X,Z),$$

$$F_2 \in \mathcal{F}_r^b(Y,Z) \text{ and } A \in \Phi^b(X,Y), \text{ imply } F_2A \in \mathcal{F}_r^b(X,Z).$$

Proof. (i) Since $A \in \Phi^b(Y,Z)$, then there exist $A_0 \in \mathcal{L}(Z,Y)$ and $K \in \mathcal{K}(Z)$ such that $AA_0 = I_Z - K$. By [24, Theorem 5.5. p. 105] we have $A_0 \in \Phi^b(Z,Y)$. Let $B \in \Phi_l^b(X,Z)$ (resp. $B \in \Phi_r^b(X,Z)$). Using the Propriety 2.3(i) we deduce that $A_0B \in \Phi_l^b(X,Y)$ (resp. $A_0B \in \Phi_r^b(X,Y)$). Then $A_0B + F_1 \in \Phi_l^b(X,Y)$ (resp. $A_0B + F_1 \in \Phi_r^b(X,Y)$). And so $AF_1 + B - KB \in \Phi_l^b(X,Y)$ (resp. $AF_1 + B - KB \in \Phi_r^b(X,Y)$). Therefore $AF_1 + B \in \Phi_l^b(X,Y)$ (resp. $AF_1 + B \in \Phi_r^b(X,Y)$).
(ii) The proof of (ii) is obtained as like as the proof of (i). □

Theorem 2.1. *Let X, Y and Z be Banach spaces.*
(i) *If the set $\Phi^b(Y,Z)$ is not empty, then*

$$F_1 \in \mathcal{F}_l^b(X,Y) \text{ and } A \in \mathcal{L}(Y,Z), \text{ imply } AF_1 \in \mathcal{F}_l^b(X,Z),$$

$$F_1 \in \mathcal{F}_r^b(X,Y) \text{ and } A \in \mathcal{L}(Y,Z), \text{ imply } AF_1 \in \mathcal{F}_r^b(X,Z).$$

(ii) *If the set $\Phi^b(X,Y)$ is not empty, then*

$$F_2 \in \mathcal{F}_l^b(Y,Z) \text{ and } A \in \mathcal{L}(X,Y), \text{ imply } F_2A \in \mathcal{F}_l^b(X,Z),$$

$$F_2 \in \mathcal{F}_r^b(Y,Z) \text{ and } A \in \mathcal{L}(X,Y), \text{ imply } F_2A \in \mathcal{F}_r^b(X,Z).$$

Remark 2.2. It follows from Definition 2.4 and the previous theorem that $\mathcal{F}_l^b(X)$ and $\mathcal{F}_r^b(X)$ are two-sided ideals of $\mathcal{L}(X)$.

Proof of Theorem 2.1. (i) Let $C \in \Phi^b(Y,Z)$ and $\lambda \in \mathbb{C}$. We denote by $A_1 = A - \lambda C$ and $A_2 = \lambda C$. For sufficiently large λ, using [24, Theorem 5.11], we have $A_1 \in \Phi^b(Y,Z)$. It follows from Proposition 2.4(i) that $A_1F_1 \in \mathcal{F}_l^b(X,Z)$ (resp. $A_1F_1 \in \mathcal{F}_r^b(X,Z)$) and $A_2F_1 \in \mathcal{F}_l^b(X,Z)$ (resp. $A_2F_1 \in \mathcal{F}_r^b(X,Z)$). This implies $A_1F_1 + A_2F_1 = AF_1 \in \mathcal{F}_l^b(X,Z)$ (resp. $A_1F_1 + A_2F_1 = AF_1 \in \mathcal{F}_r^b(X,Z)$). (ii) We can check the other results in the same way us the previous one. □

Proposition 2.5. *Let X and Y be two Banach spaces. If the set $\Phi^b(Y,Z)$ is not empty, then we have the inclusions:*

$$\mathcal{K}(X,Y) \subset \mathcal{F}_l^b(X,Y) \subset \mathcal{F}^b(X,Y),$$

$$\mathcal{K}(X,Y) \subset \mathcal{F}_r^b(X,Y) \subset \mathcal{F}^b(X,Y).$$

Proof. We will prove the first result. The same reasoning remains valid for the second one. It is obvious that $\mathcal{K}(X,Y) \subset \mathcal{F}_l^b(X,Y)$. For the second inclusion, let $F \in \mathcal{F}_l^b(X,Y)$ and consider $A \in \Phi^b(X,Y)$, then there exist $A_0 \in \mathcal{L}(Y,X)$ and $K \in \mathcal{K}(X)$ such that $A_0A = I_X - K$. So, $A_0(A+F) = I_X - K + A_0F$. It follows from Theorem 2.1 that $A_0F \in \mathcal{F}_l^b(X)$, then $A_0(A+F) \in \Phi_l^b(X)$. Using the inclusion $\Phi_l^b(X,Y) \subset \Phi_+^b(X,Y)$, we obtain $A+F \in \Phi_+^b(X,Y)$. On the other hand, consider the mapping φ defined by: $\forall\lambda \in \mathbb{C}, \varphi(\lambda) = A + \lambda F$. Note that φ is continuous and $\varphi([0,1]) \subset \Phi_+^b(X,Y)$, using Proposition 2.1, we can deduce that $i(A+F) = i(A) < \infty$. Hence $A + F \in \Phi^b(X,Y)$. □

3. Some results on perturbation theory of 2×2 matrix operator

In this section we will establish some results on perturbation theory of 2×2 matrix operator that acts on a product of Banach spaces X_1 and X_2. The following lemmas are necessary.

Lemma 3.1. *Let $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and consider the 2×2 matrix operator $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where $C \in \mathcal{L}(X_2, X_1)$. Then*

(i) If $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $M_C \in \Phi^b(X_1 \times X_2)$, for every $C \in \mathcal{L}(X_2, X_1)$.

(ii) If $A \in \Phi_+^b(X_1)$ and $B \in \Phi_+^b(X_2)$, then $M_C \in \Phi_+^b(X_1 \times X_2)$ for every $C \in \mathcal{L}(X_2, X_1)$.

(iii) If $A \in \Phi_-^b(X_1)$ and $B \in \Phi_-^b(X_2)$, then $M_C \in \Phi_-^b(X_1 \times X_2)$ for every $C \in \mathcal{L}(X_2, X_1)$.

Proof. (i) Write M_C in the form

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{3.1}$$

Since $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ are both Fredholm operators. So, M_C is a Fredholm operator, since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible for every $C \in \mathcal{L}(X_2, X_1)$.

(ii) and (iii) can be checked in the same way as (i). □

Remark 3.1. Using the same reasoning as the proof of the previous lemma we can show that:
(i) If $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $M_D := \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ is a Fredholm operator on $X_1 \times X_2$ for every $D \in \mathcal{L}(X_1, X_2)$.

(ii) If $A \in \Phi^b_+(X_1)$ and $B \in \Phi^b_+(X_2)$, then $M_D \in \Phi^b_+(X_1 \times X_2)$ for every $D \in \mathcal{L}(X_1, X_2)$.

(iii) If $A \in \Phi^b_-(X_1)$ and $B \in \Phi^b_-(X_2)$, then $M_D \in \Phi^b_-(X_1 \times X_2)$ for every $D \in \mathcal{L}(X_1, X_2)$.

Lemma 3.2. *Let $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and consider the 2×2 matrix operator $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where $C \in \mathcal{L}(X_2, X_1)$.*

(i) If $M_C \in \Phi^b_+(X_1 \times X_2)$, then $A \in \Phi^b_+(X_1)$.

(ii) If $M_C \in \Phi^b_-(X_1 \times X_2)$, then $B \in \Phi^b_-(X_2)$.

Proof. The result follows immediately from Eq. (3.1). □

Remark 3.2. (i) It follows immediately from the last Lemma that if $M_C \in \Phi^b(X_1 \times X_2)$, then $A \in \Phi^b_+(X_1)$ and $B \in \Phi^b_-(X_2)$.
(ii) Using the same reasoning as the proof of the previous lemma we can show that if the operator $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ is in $\Phi^b(X_1 \times X_2)$ for some $D \in \mathcal{L}(X_1, X_2)$, then $A \in \Phi^b_-(X_1)$ and $B \in \Phi^b_+(X_2)$.

Theorem 3.1. *Let $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ where $F_{ij} \in \mathcal{L}(X_j, X_i)$, $i, j = 1, 2$. Then*

$$F \in \mathcal{F}^b(X_1 \times X_2) \text{ if and only if } F_{ij} \in \mathcal{F}^b(X_j, X_i),\ \forall i, j = 1, 2.$$

Remark 3.3. (i) It follows from Lemma 2.1 that Theorem 3.1 remains valid if we replace $\mathcal{F}^b(X_1 \times X_2)$ by $\mathcal{F}(X_1 \times X_2)$ and $\mathcal{F}^b(X_j, X_i)$ by $\mathcal{F}(X_j, X_i)$, $i, j = 1, 2$.
(ii) It is sufficient to apply the definition of compact and strictly singular operators to verify that the result of Theorem 3.1 is true for these classes of operators. Therefore, in view of Remark 2.1 the previous theorem may be viewed as a generalization to a more large class of operators.

Proof. To prove the second implication, we consider the following decomposition,

$$F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_{22} \end{pmatrix}.$$

It is sufficient to prove that if $F_{ij} \in \mathcal{F}^b(X_j, X_i)$, $i, j = 1, 2$ then, each operator in the right hand side of the previous equality is a Fredholm perturbation on $X_1 \times X_2$. We will prove the result for example for the first operator. The proof for the other operators will be in the same way. Consider $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi^b(X_1 \times X_2)$ and denote $\widetilde{F} := \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$. It follows from [17, Theorem 12 p.159] that there exist $L_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$ and $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \in \mathcal{K}(X_1 \times X_2)$ such that

$$LL_0 = I - K \text{ on } X_1 \times X_2.$$

Then,

$$(L + \widetilde{F})L_0 = I - K + \widetilde{F}L_0 = \begin{pmatrix} I - K_{11} + F_{11}A_0 & -K_{12} + F_{11}B_0 \\ -K_{21} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}^b(X_1)$ and using Theorem 2.1(ii), we will have $I - K_{11} + F_{11}A_0 \in \Phi^b(X_1)$. This, with the fact that $I - K_{22} \in \Phi^b(X_2)$, we can deduce from Lemma 3.1(i) that $(L + \widetilde{F})L_0 - \begin{pmatrix} 0 & 0 \\ -K_{21} & 0 \end{pmatrix}$ is a Fredholm operator on $X_1 \times X_2$. The fact that K_{21} is a compact operator and $L_0 \in \Phi^b(X_1 \times X_2)$ leads, by [24, Theorem 5.13], to $L + \widetilde{F} \in \Phi^b(X_1 \times X_2)$.

Conversely, assume that $F \in \mathcal{F}^b(X_1 \times X_2)$. We will prove that $F_{11} \in \mathcal{F}^b(X_1)$. Let $A \in \Phi^b(X_1)$ and define the operator $L_1 := \begin{pmatrix} A & -F_{12} \\ 0 & I \end{pmatrix}$. It follows, from Lemma 3.1(i) that $L_1 \in \Phi^b(X_1 \times X_2)$. Thus $F + L_1 = \begin{pmatrix} A + F_{11} & 0 \\ F_{21} & I + F_{22} \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. The use of Remark 3.2(ii) leads to

$$A + F_{11} \in \Phi^b_-(X_1). \tag{3.2}$$

In the same way, we consider the Fredholm operator $\begin{pmatrix} A & 0 \\ -F_{21} & I \end{pmatrix}$ and we use Remarks 3.1(i) and 3.2(i) to deduce that

$$A + F_{11} \in \Phi^b_+(X_1). \tag{3.3}$$

It follows from Eqs. (3.2) and (3.3) that

$$F_{11} \in \mathcal{F}^b(X_1).$$

In the same way, we prove that $F_{22} \in \mathcal{F}^b(X_2)$.

Now, we will prove that $F_{12} \in \mathcal{F}^b(X_2, X_1)$ and $F_{21} \in \mathcal{F}^b(X_1, X_2)$. For this, consider $A \in \Phi^b(X_2, X_1)$ and $B \in \Phi^b(X_1, X_2)$. Then $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. Using the fact that $F_{11} \in \mathcal{F}^b(X_1)$, $F_{22} \in \mathcal{F}^b(X_2)$ and the result of the second implication, we deduce that $F + \begin{pmatrix} -F_{11} & 0 \\ 0 & -F_{22} \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2)$. Hence, $\begin{pmatrix} 0 & A + F_{12} \\ B + F_{21} & 0 \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. So, $A + F_{12} \in \Phi^b(X_2, X_1)$ and $B + F_{21} \in \Phi^b(X_1, X_2)$. $\square$

Theorem 3.2. *Let* $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ *where* $F_{ij} \in \mathcal{L}(X_j, X_i)$, $i, j = 1, 2$. *Then*
(i) $F \in \mathcal{F}_l^b(X_1 \times X_2)$ *if and only if* $F_{ij} \in \mathcal{F}_l^b(X_j, X_i)$, $\forall i, j = 1, 2$.
(ii) $F \in \mathcal{F}_r^b(X_1 \times X_2)$ *if and only if* $F_{ij} \in \mathcal{F}_r^b(X_j, X_i)$, $\forall i, j = 1, 2$.

Proof. (i) Using the same notations as in the proof of Theorem 3.1 we obtain:

$$L_0(L + \widetilde{F}) = I - K + L_0\widetilde{F} = \begin{pmatrix} I - K_{11} + A_0F_{11} & -K_{12} \\ -K_{21} + C_0F_{11} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}_l^b(X_1)$ and using Theorem 2.1(i), we deduce that $I - K_{11} + A_0F_{11} \in \Phi_l^b(X_1)$. So, there exist an operator $H \in \mathcal{L}(X_1 \times X_2)$ and $K_0 \in \mathcal{K}(X_1)$ such that $H(I - K_{11} + A_0F_{11}) = I - K_0$. Therefore,

$$\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) = I - \begin{pmatrix} K_0 & HK_{12} \\ K_{21} & K_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C_0F_{11} & 0 \end{pmatrix}.$$

Using Theorem 2.1(i), Proposition 2.5(i) and Theorem 3.1 we obtain $\begin{pmatrix} 0 & 0 \\ C_0F_{11} & 0 \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2)$, and so, $\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) \in \Phi^b(X_1 \times X_2)$, then there exist $L_1 \in \mathcal{L}(X_1 \times X_2)$ and $\widetilde{K} \in \mathcal{K}(X_1 \times X_2)$ such that $L_1 \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) = I - \widetilde{K}$, which implies that $\widetilde{F} \in \mathcal{F}_b^l(X_1 \times X_2)$.
(ii) We prove this assertion in the same way as in (i). $\square$

Remark 3.4. The following questions remain open:
(i) $F \in \mathcal{F}_+^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_+^b(X_j, X_i)$, $\forall i, j = 1, 2$.
(ii) $F \in \mathcal{F}_-^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_-^b(X_j, X_i)$, $\forall i, j = 1, 2$.

4. The M-essential spectra of 2×2 matrix operator

The purpose of this section is to discuss the M-essential spectra of the 2×2 matrix operator L, closure of L_0 that acts on the Banach space $X \times Y$ where M is a bounded operator formally defined on the product space $X \times Y$ by a matrix

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and L_0 is given by

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The operator A acts on X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts on X (resp. on Y).

In what follows, we will assume that the following conditions hold:

(H_1) A is a closed, densely defined linear operator on X with nonempty M_1-resolvent set $\rho_{M_1}(A)$.

(H_2) The operator B is densely defined linear operator on X and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $(A - \mu M_1)^{-1}B$ is closable. (In particular, if B is closable then $(A - \mu M_1)^{-1}B$ is closable).

(H_3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $C(A - \mu M_1)^{-1}$ is bounded.

(H_4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in Y, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $D - C(A - \mu M_1)^{-1}B$ is closable, we will denote by $S(\mu)$ the closure of the operator $D - (C - \mu M_3)(A - \mu M_1)^{-1}(B - \mu M_2)$.

Remark 4.1. (i) It follows from the closed graph theorem that the operator $G(\mu) := (A - \mu M_1)^{-1}(B - \mu M_2)$ is bounded on Y.

(ii) We emphasize that neither the domain of $S(\mu)$ nor the property of being closable depend on μ. Indeed, consider $\lambda, \mu \in \rho_{M_1}(A)$, then we have:

$$S(\lambda) - S(\mu) = (\lambda - \mu)\left[M_3 G(\mu) + F(\lambda)M_2 + F(\lambda)M_1 G(\mu)\right], \tag{4.1}$$

where $F(\lambda) = (C - \lambda M_3)(A - \lambda M_1)^{-1}$. Since the operators $F(\lambda)$ and $G(\mu)$ are bounded, then the difference $S(\lambda) - S(\mu)$ is bounded. Therefore neither the domain of $S(\mu)$ nor the property of being closable depend on μ.

We recall the following result established in [8] which describes the closure of the operator L_0.

Theorem 4.1. *[8] Let conditions $(H_1)-(H_3)$ be satisfied and the lineal $\mathcal{D}(B)\cap\mathcal{D}(D)$ be dense in X. Then the operator L_0 is closable if and only if the operator $D - C(A - \mu M_1)^{-1}B$ is closable in X, for some $\mu \in \varrho_{M_1}(A)$. Moreover, the closure L of L_0 is given by*

$$L = \mu M + \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} A - \mu M_1 & 0 \\ 0 & S(\mu) - \mu M_4 \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \qquad (4.2)$$

Lemma 4.1. *(i) If $M_3 \in \mathcal{F}^b(X,Y)$ and $F(\lambda) \in \mathcal{F}^b(X,Y)$, for some $\lambda \in \rho_{M_1}(A)$, then $F(\lambda) \in \mathcal{F}^b(X,Y)$ for all $\lambda \in \rho_{M_1}(A)$.*

(ii) If $M_2 \in \mathcal{F}^b(Y,X)$ and if $G(\lambda) \in \mathcal{F}^b(Y,X)$, for some $\lambda \in \rho_{M_1}(A)$, then $G(\lambda) \in \mathcal{F}^b(Y,X)$ for all $\lambda \in \rho_{M_1}(A)$.

(iii) If $F(\lambda)$, $G(\lambda)$, M_2 and M_3 are Fredholm perturbations, for some $\lambda \in \rho_{M_1}(A)$, then $\sigma_{e_i,M_4}(S(\lambda))$ does not depend on $\lambda \in \rho_{M_1}(A)$, for $i = 1, ..., 6$.

Proof. (i) The result follows from the identity

$$F(\lambda) - F(\mu) = (\lambda - \mu)\big[F(\lambda)M_1 - M_3\big](A - \mu M_1)^{-1}, \text{ for all } \lambda \text{ and } \mu \in \rho_{M_1}(A).$$

(ii) The result follows from the identity

$$G(\lambda) - G(\mu) = (\lambda - \mu)(A - \lambda M_1)^{-1}\big[M_1 G(\mu) - M_2\big], \text{ for all } \lambda \text{ and } \mu \in \rho_{M_1}(A).$$

(iii) The result of this assertion follows from Eq. (4.1). ☐ In the sequel, we will denote the complement of a subset $\Omega \subset$ C by ${}^C\Omega$.

Theorem 4.2. *Let L_0 be the 2×2 matrix operator satisfying conditions $(H_1)-(H_4)$. If M_2 and M_3 are Fredholm perturbations and if for some (hence for all) $\mu \in \rho_{M_1}(A)$, $F(\mu)$ and $G(\mu)$ are Fredholm perturbations, then*

$$\sigma_{e_4,M}(L) = \sigma_{e_4,M_1}(A) \cup \sigma_{e_4,M_4}(S(\mu)).$$

and

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)).$$

Moreover, if ${}^C\sigma_{e_4,M_1}(A)$ is connected, then

$$\sigma_{e_5,M}(L) = \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)).$$

If in addition, ${}^C\sigma_{e_5,M}(L)$ is connected, $\rho_M(L) \neq \emptyset$, ${}^C\sigma_{e_5,M_4}(S(\mu))$ is connected and $\rho_{M_4}(S(\mu)) \neq \emptyset$, then

$$\sigma_{e_6,M}(L) = \sigma_{e_6,M_1}(A) \cup \sigma_{e_6,M_4}(S(\mu)).$$

Proof. Let $\mu \in \rho_{M_1}(A)$ be such that the operators $F(\mu)$ and $G(\mu)$ are Fredholm perturbations and set $\lambda \in \mathbb{C}$. While writing $\lambda M - L = \mu M - L + (\lambda - \mu)M$, using the relation (4.2) we have

$$\lambda M - L = UV(\lambda)W - (\lambda - \mu) \begin{pmatrix} 0 & M_1 G(\mu) - M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1 G(\mu) \end{pmatrix} \qquad (4.3)$$

where $U=\begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}$, $W=\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$ and $V(\lambda)=\begin{pmatrix} \lambda M_1 - A & 0 \\ 0 & \lambda M_4 - S(\mu) \end{pmatrix}$. Since the operators $F(\mu)$, $G(\mu)$, M_2 and M_3 are Fredholm perturbations, then by Theorem 3.1 the second operator in the right hand side of Eq.(4.3) is a Fredholm perturbation. So $\lambda M - L$ is a Fredholm operator if and only if $UV(\lambda)W$ is a Fredholm operator. Now, observe that the operators U and W are bounded and have bounded inverse, hence the operator $UV(\lambda)W$ is a Fredholm operator if and only if $V(\lambda)$ has this property if and only if $\lambda M_1 - A$ (resp. $\lambda M_4 - S(\mu)$) is a Fredholm operator on X (resp. on Y) and

$$i(\lambda M - L) = i(\lambda M_1 - A) + i(\lambda M_4 - S(\mu)). \tag{4.4}$$

Therefore,

$$\sigma_{e_4,M}(L) = \sigma_{e_4,M_1}(A) \cup \sigma_{e_4,M_4}(S(\mu))$$

and

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)). \tag{4.5}$$

Suppose now that ${}^{C}\sigma_{e_4,M_1}(A)$ is connected. By assumption (H_1), $\rho_{M_1}(A)$ is not empty. Let $\alpha \in \rho_{M_1}(A)$, then, $\alpha M_1 - A \in \Phi(X)$ and $i(\alpha M_1 - A) = 0$. Since $\rho_{M_1}(A) \subseteq \rho_{4,M_1}(A)$ and by Proposition 2.1, $i(\lambda M_1 - A)$ is constant on any component of $\Phi_{M_1,A}$, then $i(\lambda M_1 - A) = 0$ for all $\lambda \in \rho_{4,M_1}(A)$. It follows, immediately, from Eqs (4.4) and (4.5) that

$$\sigma_{e_5,M}(L) = \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)). \tag{4.6}$$

Assume further, that ${}^{C}\sigma_{e_5,M_1}(A)$ is connected. Then, by Lemma 2.1 in [15] and using Eq. 4.6 we have

$$\sigma_{e_6,M}(L) = \sigma_{e_6,M_1}(A) \cup \sigma_{e_6,M_4}(S(\mu)). \qquad \square$$

In the sequel we will denote, for $\mu \in \varrho_{M_1}(A)$, by $M(\mu)$ the following operator

$$M(\mu) = \begin{pmatrix} 0 & M_1 G(\mu) - M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1G(\mu) \end{pmatrix}.$$

Theorem 4.3. *(i) If the operator $M(\mu) \in \mathcal{F}_+(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then*

$$\sigma_{e_1,M}(L) = \sigma_{e_1,M_1}(A) \cup \sigma_{e_1,M_4}(S(\mu)).$$

(ii) If the operator $M(\mu) \in \mathcal{F}_-(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then

$$\sigma_{e_2,M}(L) = \sigma_{e_2,M_1}(A) \cup \sigma_{e_2,M_4}(S(\mu)).$$

(iii) If $M(\mu) \in \mathcal{F}_+(X \times Y) \cap \mathcal{F}_-(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then

$$\begin{aligned} \sigma_{e_3,M}(L) &= \sigma_{e_3,M_1}(A) \cup \sigma_{e_3,M_4}(S(\mu)) \cup [\sigma_{e_2,M_1}(A) \cup \sigma_{e_1,M_4}(S(\mu))] \\ &\quad \cup [\sigma_{e_1,M_1}(A) \cup \sigma_{e_2,M_4}(S(\mu))]. \end{aligned}$$

Proof. The assertions (i) and (ii) follow immediately from Eq. (4.3).
The assertion (iii) is an immediate consequence of (i) and (ii). $\square$

Remark 4.2. Theorems (4.2) and (4.3) generalize the Theorem (3.2) in [16].

5. Application to two-group transport operators

The aim of this section is to apply the obtained results to study the M-essential spectra of a class of linear two-group transport operators on L_p-spaces, $1 \leq p < \infty$, with abstract boundary conditions.

Let

$$X_p := L_p((-a,a) \times (-1,1); dxdv), \quad a > 0,\ 1 \leq p < \infty.$$

We consider the following two-group transport operators with abstract boundary conditions:

$$A_H = T_H + K$$

where

$$T_H\psi = \begin{pmatrix} T_{H_1} & 0 \\ 0 & T_{H_2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with $K_{ij}, i,j = 1,2$, are bounded linear operators defined on X_p by

$$\left\{ \begin{array}{llll} K_{ij}: & X_p & \longrightarrow & X_p \\ & u & \longrightarrow & K_{ij}u(x,v) = \displaystyle\int_{-1}^{1} \kappa_{ij}(x,v,v')u(x,v')dv' \end{array} \right. \tag{5.1}$$

and the kernels $\kappa_{ij} : (-a,a) \times (-1,1) \times (-1,1) \longrightarrow \mathrm{R}$ are assumed to be measurable. Each operator T_{H_j}, $j = 1,2$, is defined by

$$\left\{ \begin{array}{l} T_{H_j} : \mathcal{D}(T_{H_j}) \subset X_p \longrightarrow X_p, \\ \qquad\qquad \varphi \qquad\quad \longrightarrow (T_{H_j}\varphi)(x,v) = -v\dfrac{\partial\varphi}{\partial x}(x,v) - \sigma_j(v)\varphi(x,v), \\ \\ \mathcal{D}(T_{H_j}) = \{\varphi \in W \text{ such that } \varphi^i = H_j\varphi^o\}, \end{array} \right.$$

where W is the space defined by

$$W = \{\varphi \in X_p \text{ such that } v\frac{\partial\varphi}{\partial x} \in X_p\}$$

and $\sigma_j(.) \in L^\infty(-1,1)$. φ^o, φ^i represent the outgoing and the incoming fluxes related by the boundary operator H_j ("o" for the outgoing and "i" for the incoming) and given by

$$\left\{ \begin{array}{ll} \varphi^i(v) = \varphi(-a,v), & v \in (0,1), \\ \varphi^i(v) = \varphi(a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(-a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(a,v), & v \in (0,1). \end{array} \right.$$

We denote by X_p^o and X_p^i the following boundary spaces:

$X_p^o := L_p\,[\{-a\}\times(-1,0);|v|dv]\times L_p\,[\{a\}\times(0,1);|v|dv] := X_{1,p}^o\times X_{2,p}^o$
equipped with the norm

$$\begin{aligned}\|u^o, X_p^o\| &:= \left(\|u_1^o, X_{1,p}^o\|^p + \|u_2^o, X_{2,p}^o\|^p\right)^{\frac{1}{p}}\\ &= \left[\int_{-1}^{0}|u(-a,v)|^p|v|dv + \int_0^1 |u(a,v)|^p|v|dv\right]^{\frac{1}{p}},\end{aligned}$$

and

$X_p^i := L_p\,[\{-a\}\times(0,1);|v|dv]\times L_p\,[\{a\}\times(-1,0);|v|dv] := X_{1,p}^i\times X_{2,p}^i$
equipped with the norm

$$\begin{aligned}\|u^i, X_p^i\| &:= \left(\|u_1^i, X_{1,p}^i\|^p + \|u_2^i, X_{2,p}^i\|^p\right)^{\frac{1}{p}}\\ &= \left[\int_0^{1}|u(-a,v)|^p|v|dv + \int_{-1}^0 |u(a,v)|^p|v|dv\right]^{\frac{1}{p}}.\end{aligned}$$

It is well known that any function u in W possesses traces on the spacial boundary $\{-a\}\times(-1,0)$ and $\{a\}\times(0,1)$ which respectively belong to the spaces X_p^o and X_p^i (see, for instance, [7] or [11]). they are denoted, respectively, by u^o and u^i.

It is clear that the operator A_H is defined on $\mathcal{D}(T_{H_1})\times\mathcal{D}(T_{H_2})$. We will denote the operator A_H by

$$A_H := \begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},$$

where

$$\begin{cases} A_{11} = T_{H_1} + K_{11},\\ A_{12} = K_{12},\\ A_{21} = K_{21},\\ A_{22} = T_{H_2} + K_{22}.\end{cases}$$

The object of this part is to determine the M-essential spectra of the operator A_H where M is the following operator

$$M = \begin{pmatrix} M_1 & M_2\\ M_3 & M_4\end{pmatrix}$$

with $M_i,\ i = 1,4$ are defined by

$$\begin{cases} M_i: & X_p & \longrightarrow & X_p\\ & \varphi & \longrightarrow & (M_i\varphi)(x,v) = \eta_i(v)\varphi(x,v)\end{cases}$$

where $\eta_i(.)\in\mathcal{L}^\infty(-1,1)$ and M_2, M_3 are in $\mathcal{F}(X_p)$.

To verify the hypotheses of Theorem 4.2, we shall first determine the expression of the M_1-resolvent of the operator T_{H_1}. Let $\varphi\in X_p,\ \lambda\in\mathbb{C}$ and consider the M_1-resolvent equation for T_{H_1}

$$(\lambda M_1 - T_{H_1})\psi_1 = \varphi, \tag{5.2}$$

where the unknown ψ_1 must be in $\mathcal{D}(T_{H_1})$. Let

$$\lambda_j^* = \text{ ess-inf } \sigma_j(v), \qquad j = 1,2;$$

$$\mu_j^* = \text{ ess-inf } \eta_j(v), \qquad j = 1, 2;$$

we suppose that $\mu_j^* > 0$, $\ j = 1, 2$ and let

$$\lambda_0^j := \begin{cases} -\lambda_j^*, & \text{if } \|H_j\| \leq 1 \\ -\dfrac{\lambda_j^*}{\mu_j^*} + \dfrac{1}{2a\mu_j^*} \log(\|H_j\|), & \text{if } \|H_j\| > 1. \end{cases}$$

Therefore, for $\lambda \in \mathbb{C}$ such that $\mu_1^* Re\lambda + \lambda_1^* > 0$, the solution of Eq. (5.2) is formally given by

$$\psi_1(x, v) = \begin{cases} \psi_1(-a, v) e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a+x|}{|v|}} \\ + \dfrac{1}{|v|} \displaystyle\int_{-a}^{x} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|x-x'|}{|v|}} \varphi(x', v) dx', \quad 0 < v < 1, \\ \psi_1(a, v) e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a-x|}{|v|}} \\ + \dfrac{1}{|v|} \displaystyle\int_{x}^{a} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|x-x'|}{|v|}} \varphi(x', v) dx', \quad -1 < v < 0. \end{cases} \tag{5.3}$$

Accordingly, $\psi_1(a, v)$ and $\psi_1(-a, v)$ are given by

$$\psi_1(a, v) = \psi_1(-a, v) e^{-2a\frac{(\lambda\eta_1(v) + \sigma_1(v))}{|v|}}$$

$$+ \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a-x|}{|v|}} \varphi(x, v) dx, \quad 0 < v < 1, \tag{5.4}$$

$$\psi_1(-a, v) = \psi_1(a, v) e^{-2a\frac{(\lambda\eta_1(v) + \sigma_1(v))}{|v|}} \tag{5.5}$$

$$+ \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a+x|}{|v|}} \varphi(x, v) dx, \quad -1 < v < 0. \tag{5.6}$$

For the clarity of our subsequent analysis, we introduce the following bounded operators:

$$\begin{cases} M_\lambda : X_p^i \to X_p^o, \quad M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \quad \text{with} \\ M_\lambda^+ u(-a, v) := u(-a, v) e^{-2a\frac{(\lambda\eta_1(v) + \sigma_1(v))}{|v|}}, \quad 0 < v < 1, \\ M_\lambda^- u(a, v) := u(a, v) e^{-2a\frac{(\lambda\eta_1(v) + \sigma_1(v))}{|v|}}, \quad -1 < v < 0, \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \to X_p, \quad B_\lambda u := \chi(-1, 0)(v) B_\lambda^- u + \chi(0, 1)(v) B_\lambda^+ u) \quad \text{with} \\ B_\lambda^+ u(x, v) := u(-a, v) e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a+x|}{|v|}}, \quad 0 < v < 1, \\ B_\lambda^- u(x, v) := u(a, v) e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a-x|}{|v|}}, \quad -1 < v < 0, \end{cases}$$

$$\begin{cases} G_\lambda : X_p \to X_p^o, \quad G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \quad \text{with} \\ G_\lambda^+ \varphi(-a, v) := \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a-x|}{|v|}} \varphi(x, v) dx, \quad 0 < v < 1, \\ G_\lambda^- \varphi(a, v) := \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v) + \sigma_1(v))|a+x|}{|v|}} \varphi(x, v) dx, \quad -1 < v < 0, \end{cases}$$

and finally, we consider

$$\begin{cases} C_\lambda : X_p \to X_p, \quad C_\lambda\varphi := \chi(-1,0)C_\lambda^-\varphi + \chi(0,1)C_\lambda^+\varphi \quad \text{with} \\ C_\lambda^+\varphi(x,v) := \frac{1}{|v|}\int_{-a}^{x} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)dx', \quad 0 < v < 1, \\ C_\lambda^-\varphi(x,v) := \frac{1}{|v|}\int_{x}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)dx', \quad -1 < v < 0, \end{cases}$$

where $\chi_{(0,1)}(.)$ and $\chi_{(-1,0)}(.)$ denote the characteristic functions of the intervals $(-1,0)$ and $(0,1)$, , respectively. The operators M_λ, B_λ, G_λ and C_λ are bounded by : $e^{-2a\mu^* Re\lambda}$, $(p\mu^* Re\lambda)^{-1/p}$, $(\mu^* Re\lambda)^{-1/q}$, respectively, where q denotes the conjugate of p and $(\mu^* Re\lambda)^{-1}$.

Lemma 5.1. (i) *If* $\kappa_{ij}(x,v,v')$ *defines a regular operator, then* $(\lambda M_1 - T_{H_1})^{-1}K_{ij}$ *is compact on* X_p, *for* $1 < p < \infty$ *and weakly compact on* X_1, $i,j = 1,2$.
(ii) *If* $\kappa_{ij}(x,v,v')/|v'|$ *defines a regular operator, then* $K_{ij}(\lambda M_1 - T_{H_1})^{-1}$ *is weakly compact on* X_1, $i,j = 1,2$.

Proof. (i) This assertion was proved in [15].
(ii) The proof of this assertion is a straightforward adaption from Lemma 4.2 in [16]. □

Theorem 5.1. *If* $\kappa_{21}(x,v,v')$ *(resp.* $\kappa_{21}(x,v,v')/|v'|$*) defines a regular operator, then the operators* $F(\lambda) := (K_{21} - \lambda M_3)(A_{11} - \lambda M_1)^{-1}$ *and* $G(\lambda) := (A_{11} - \lambda M_1)^{-1}(K_{12} - \lambda M_2)$ *are Fredholm perturbations on* X_p, $1 \le p < \infty$.

Proof. It follows from Remark 3.1.(ii) in [15] that there exists $\lambda \in \rho_{M_1}(T_{H_1})$ such that
$r_\sigma\left((T_{H_1} - \lambda M_1)^{-1}K_{11}\right) < 1$. For such λ, the equation

$$(K_{11} + T_{H_1} - \lambda M_1)\varphi = \psi$$

may be transformed into

$$\left((T_{H_1} - \lambda M_1)^{-1}K_{11} - I\right)\varphi = (T_{H_1} - \lambda M_1)^{-1}\psi.$$

Then, by the fact that $r_\sigma\left((T_{H_1} - \lambda M_1)^{-1}K_{11}\right) < 1$, we obtain

$$(A_{11} - \lambda M_1)^{-1} = \sum_{n\ge 0}[(T_{H_1} - \lambda M_1)^{-1}K_{11}]^n(T_{H_1} - \lambda M_1)^{-1}.$$

So,

$$F(\lambda) = K_{21}\sum_{n\ge 0}\left[(T_{H_1} - \lambda M_1)^{-1}K_{11}\right]^n(T_{H_1} - \lambda M_1)^{-1} - \lambda M_3(A_{11} - \lambda M_1)^{-1}.$$

Since $M_3 \in \mathcal{F}(X_p)$ and the use of Lemma 5.1 allows us to conclude that $F(\lambda) \in \mathcal{F}(X_p)$.
The same reasoning allows us to prove that $G(\lambda) \in \mathcal{F}(X_p)$. □

Now, we are ready to express the M-essential spectra of two-group transport operators with general boundary conditions.

Theorem 5.2. *If the operators $H_j \in \mathcal{F}(X_p)$, $j = 1, 2$, $1 \leq p < \infty$ and the operators K_{11}, K_{22}, K_{12} are regular and if in addition $\kappa_{21}(x, v, v')$ (resp. $\kappa_{21}(x, v, v')/|v'|$) defines a regular operator on X_p, for $1 < p < \infty$ (resp. on X_1), then*

$$\sigma_{e_i,M}(A_H) = \{\lambda \in \mathrm{C} \textit{ such that } Re\lambda \leq -min(\lambda_1^*, \lambda_2^*)\}, \quad \textit{for } i = 1, ..., 6.$$

Proof. Let $\lambda \in \rho_{M_1}(T_{H_1})$ such that $r_\sigma(\lambda M_1 - T_{H_1})K_{11} < 1$, then

$$(\lambda M_1 - A_{11})^{-1} - (\lambda M_1 - T_{H_1})^{-1} = \sum_{n\geq 1} [(\lambda M_1 - T_{H_1})^{-1} K_{11}]^n (\lambda M_1 - T_{H_1})^{-1}.$$

Since K_{11} is regular, then it follows from Lemma 5.1 that the operator $(\lambda M_1 - A_{11})^{-1} - (\lambda M_1 - T_{H_1})^{-1}$ is compact on X_p, for $1 < p < \infty$ and weakly compact on X_1, the use of [15, Theorem 3.3] leads to

$$\sigma_{e_i,M_1}(A_{11}) = \sigma_{e_i,M_1}(T_{H_1}) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_1^*}{\mu_1^*}\right\}, \quad i = 1, ..., 6. \quad (5.7)$$

Let $\mu \in \rho_{M_1}(A_{11})$. The operator $S(\mu)$ is given by

$$S(\mu) = A_{22} - K_{21}G(\mu).$$

By Lemma 5.1, The operator $K_{21}G(\mu)$ is compact on X_p, for $1 < p < \infty$, and weakly compact on X_1, then it follows from Proposition 2.2 that $\sigma_{e_i,M_4}(S(\mu)) = \sigma_{e_i,M_4}(A_{22})$, $i = 1, ..., 6$. By the same reasoning, we have

$$\sigma_{e_i,M_4}(S(\mu)) = \sigma_{e_i,M_1}(A_{22}) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_2^*}{\mu_2^*}\right\}, \quad i = 1, ..., 6. \quad (5.8)$$

Applying Theorem 4.2 and using Eqs (5.7) and (5.8), we get

$$\sigma_{e_i,M}(A_H) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\min\left(\frac{\lambda_1^*}{\mu_1^*}, \frac{\lambda_2^*}{\mu_2^*}\right)\right\}, \quad i = 1, ..., 6. \qquad \square$$

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Measures of Growth and Approximation of Entire Harmonic Functions in n-Dimensional Space in Some Banach Spaces

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Abstract: The relationship between the classical order and type of an entire harmonic function in space $\mathbb{R}^n, n \geq 3$, and the rate of its best harmonic polynomial approximation for some Banach spaces of functions harmonic in the ball of radius R has been studied.

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Keywords and Phrases: Entire harmonic function, Approximation errors, Banach spaces, order and type, Gegenbauer polynomials and spherical harmonics.

1. Introduction

Several authors like Vakarchuk [24], Vakarchuk and Zhir [25,26], Srivastava and Kumar [20], Harfaoui [7] and others have studied the growth parameters of an entire function in terms of coefficients occurring in its Taylor series expansion and polynomial approximation errors in some Banach spaces. Since entire harmonic functions play an important role in physics and mechanics to describe different stationary processes and in mathematical research, it is significant to study the growth characteristics order and type of entire harmonic functions in terms of coefficients occurring in its Fourier-Laplace series [23] and harmonic polynomial approximation errors in space $\mathbb{R}^n, n \geq 3$ in some Banach spaces. To the best of our knowledge this study has not been done so far. In this paper our aim is to bridge this gap.
A number of papers [3,4,10-17,19,21] were devoted to establishing a relation between

the growth of entire harmonic functions in $\mathbb{R}^n, n \geq 3$ and the behavior of expansion coefficients, spherical harmonics and harmonic polynomial approximation errors. In particular, when we discuss time dependent problems in $\mathbb{R}^3$ it leads to study the harmonic functions in $\mathbb{R}^4$. Therefore, to study the entire harmonic functions in $\mathbb{R}^n, n \geq 3$ is reasonable.

Let $x \in \mathbb{R}^n (n \geq 3)$ be an arbitrary point where $x = (x_1, x_2, \dots, x_n)$ and put $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. The set of all non-constant entire harmonic functions on $\mathbb{R}^n$ is denoted by H. For each $u \in H, r > 0$, the Fourier-Laplace series expansion of u be given as [23]

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x; u) r^k,$$

where $x \in S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ a unit sphere in $\mathbb{R}^n$ centered at the origin and

$$\begin{aligned} Y^{(k)}(x; u) =& a_1^{(k)} Y_1^{(k)}(x) + a_2^{(k)} Y_2^{(k)}(x) + \cdots + a_{\gamma_k}^{(k)} Y_{\gamma_k}^{(k)}(x), \\ a_j^{(k)} &= (u, Y_j^{(k)}) = \frac{\Gamma(n/2)}{2(\pi)^{\frac{n}{2}}} \int_{S^n} u(x) Y_j^{(k)}(x) dS, j = \overline{1, \gamma_k}, \\ \gamma_k &= \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}. \end{aligned}$$

Here dS is the element of the surface area on the sphere S^n, $(u, Y_j^{(k)})$ is the scalor product in $L^2(S^n)$ and $Y^{(k)}$ is a spherical harmonic of degree $k, k \in Z_+ = \{0, 1, 2, \dots,\}$ on the unit sphere $S^n (n \geq 2)$ [22].

Let $B_R^n = \{y \in \mathbb{R}^n : |y| \leq R\}$ be the ball of radius R in space $\mathbb{R}^n, n \geq 3$ centered at the origin, and $\overline{B_R^n}$ be the closure of B_R^n. We denote H_R, the class of harmonic functions in B_R^n and continuous on $\overline{B_R^n}, 0 < R < \infty$.
We now consider some of the Banach spaces.

1. The space B of functions harmonic in the ball B_R^n and continuous on $\overline{B_R^n}$ i.e., $u \in H_R$ with norm $\|u\| = \max_{y \in \overline{B_R^n}} |u(y)| < \infty$.

2. The Hardy spaces $H_p, p \geq 1$, of functions harmonic in the ball B_R^n with norm

$$\|u\|_{H_p} = \sup_{0<r<R} M_p(r; u), M_p(r; u) = (\frac{1}{(2\pi)^n} \int_{T^n} |u(re^{it}x)|^p dt)^{\frac{1}{p}}, p \in [1, \infty),$$

   where $T^n = \{x \in \mathbb{R}^n : 0 \leq x_j \leq 2\pi, j = \overline{1, n}\}$,
   $\|u\| = \sup_{y \in B_R^n} |u(y)|, p = \infty$.

3. The Bergman spaces H'_p of functions harmonic in the ball B_R^n for $p \in [1, \infty)$ with the norm

$$\|u\|_{H'_p} = (\frac{1}{(\pi)^n} \int_{S^n} |u(rx)|^p dx)^{\frac{1}{p}}.$$

4. The spaces $A_p, p \in (0,1)$ of functions harmonic in the ball B_R^n with norm

$$\|u\|_{A_p} = \int_{S^n} (\frac{R-r}{R})^{\frac{1}{p}-2} M_1(r;u)dr.$$

5. The spaces $B_{p,q,\lambda}, 0 < p < q \le \infty, \lambda > 0$, of functions harmonic in the ball B_R^n with the norm

$$\|u\|_{p,q,\lambda} = \{\int_{S^n} (\frac{R-r}{R})^{\lambda(\frac{1}{p}-\frac{1}{q})-1} M_q^{\lambda}(r;u)dr\}^{\frac{1}{\lambda}}, \lambda < \infty,$$

and

$$\|u\|_{p,q,\infty} = \sup_{0<r<R} \{(\frac{R-r}{R})^{(\frac{1}{p}-\frac{1}{q})} M_q(r;u)\}, \lambda = \infty,$$

for $\min(q,\lambda) \ge 1, B_{p,q,\lambda}$ are Banach spaces.

We denote a Banach space X formed by the functions harmonic in B_R^n with finite norm $\|.\|$ given by (1-5).
An approximation error of function $u \in H_R$ by harmonic polynomials $P \in \Pi_k$ is defined as

$$E_R^k(u) = \inf\{\|u(y) - P(y)\|, y \in \overline{B_R^n},$$

where Π_k be a set of harmonic polynomials of degree not exceeding k.
The relationship between the order and type of an entire function f in terms of the sequence $E_R^k(f)$ in the space H_2' were obtained in [19] and for the spaces $H_p', p \ge 1$ were studied by Ibragimov and Shikhaliev [8,9]. The spaces $A_p, p \in (0,1)$ of functions analytic in the unit disk were first studied by Hardy and Littlewood [6] and later by Romberg, Duren and Shields [2]. The spaces $B_{p,q,\lambda}, 0 < p < q \le \infty, \lambda > 0$ were considered in [5,6]. The order and type of entire functions in terms of approximation errors $E_R^k(f)$ in the spaces $B_{p,q,\lambda}$ were obtained by Vakarchuk [24].

2. Auxiliary Results

Lemma 2.1. *Let $u \in X$ and $u(\tau x) = \sum_{k=0}^{\infty} Y^k(x;u)\tau^k, 0 < \tau < R$, be an entire harmonic function in $\mathbb{R}^n$. Then*

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_X}{R^k}\}^{\frac{1}{k}} = 1. \tag{2.1}$$

Proof. For an entire harmonic function u in the space B and $H_p, 0 < p \le \infty$, respectively, the quantity $\|\tau^k\|_X, k \in \mathbb{Z}_+$ is

$$\|\tau^k\|_X = R^k$$

it gives (2.1).
In the space $X = H'_p, p \geq 1$, we have

$$\frac{\|\tau^k\|_{H'_p}}{R^k} = (kp+2)^{\frac{1}{p^2}} \leq \{kp(1+\frac{2}{kp})\}^{\frac{1}{p^2}}, k \geq 0 \tag{2.2}$$

or

$$\frac{\|\tau^k\|_{H'_p}}{R^k} \leq \chi_{H'_p} k^{\frac{1}{p^2}} \tag{2.3}$$

where $\chi_{H'_p} = p^{\frac{1}{p^2}}(1+\frac{1}{p})^{\frac{1}{p^2}}$.
From (2.3) we obtain the following upper bound

$$\lim_{k\to\infty}\{\frac{\|\tau^k\|_{H'_p}}{R^k}\}^{\frac{1}{k}} \leq 1. \tag{2.4}$$

For lower bound using (2.2) and we get

$$\frac{\|\tau^k\|_{H'_p}}{R^k} \geq p^{\frac{1}{p^2}} k^{\frac{1}{p^2}} \geq (pk)^{\frac{1}{p^2}}$$

or

$$\lim_{k\to\infty}\{\frac{\|\tau^k\|_{H'_p}}{R^k}\}^{\frac{1}{k}} \geq 1. \tag{2.5}$$

Combining (2.4) and (2.5) we get the required result.
In the space $X = A_p, 0 < p < 1$, we have

$$\frac{\|\tau^k\|_{A_p}}{R^k} = (2\pi)^{-\frac{1}{p}}(B(kp+1;\frac{1}{p}-1))^{-\frac{1}{p^2}}. \tag{2.6}$$

The right hand side of (2.6) can be estimated by using the relation between the Euler integral of the first kind $B(a,b)$ and Γ function for $a, b > 0$,

$$B(a,b) = \frac{\Gamma a \Gamma b}{\Gamma(a+b)}$$

and the asymptotic relation

$$\frac{\Gamma(\xi+s_1)}{\Gamma(\xi+s_2)} = \xi^{s_1-s_2}(1+\frac{(s_1-s_2)(s_1+s_2-1)}{2x} + o(|\xi^{-2}|)$$

where $|\xi| \geq 1, \xi \in \mathbb{R}$ and s_1 and s_2 are arbitrary fixed real numbers.
Set $\xi = kp, s_1 = \frac{1}{p}$ and $s_2 = 1$, for sufficiently large k, for $k \geq 1$ in above relations, we get

$$\begin{aligned}
\frac{\|\tau^k\|_{A_p}}{R^k} &= [\frac{(2\pi)^{-p}\Gamma(kp+\frac{1}{p})}{\Gamma^{\frac{1}{p^2}}(\frac{1}{p}-1)(kp+1)}]^{\frac{1}{p^2}} \\
&= \frac{(2\pi)^{-p}p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)}(k(1+\frac{(\frac{1}{p}-1)\frac{1}{p}}{2kp} + O(k^{-2}p^2)))^{\frac{1}{p^2}} \\
&\leq \chi_{A_p} k^{\frac{1}{p^2}},
\end{aligned}$$

where

$$\chi_{A_p} = \frac{(2\pi)^{-p} p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)} (1 + (\frac{1}{p}-1)\frac{1}{p} + A)^{\frac{1}{p^2}}$$

here A is an absolute constant independent of k.

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{A_p}}{R^k}\}^{\frac{1}{k}} \leq 1. \tag{2.7}$$

For lower bound we have

$$\frac{\|\tau^k\|_{A_p}}{R^k} \geq \frac{(2\pi)^{-p} p^{\overline{\frac{1}{p^2(\frac{1}{p}-1)}}}}{(\frac{1}{p}-1)} k^{\frac{1}{p^2}} \geq \{\frac{k}{\Gamma(\frac{1}{p}-1)}\}^{\frac{1}{p^2}}$$

or

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{A_p}}{R^k}\}^{\frac{1}{k}} \geq 1. \tag{2.8}$$

Using (2.7) and (2.8) we get the required result.
Following on the lines of [26] for single complex variable we obtain for the space $X = B_{p,q,\lambda}, 0 < p < q \leq \infty, 0 < \lambda \leq \infty$ that

$$\lim_{k\to\infty} \{\frac{\|\tau^k\|_{B_{p,q,\lambda}}}{R^k}\}^{\frac{1}{k}} = 1.$$

□

Lemma 2.2. *Let $u \in X$ and let*

$$u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x;u)\tau^k \quad \textit{in space} \quad \mathbb{R}^n, n \geq 3, 0 < \tau < R.$$

Then

$$|Y^{(k)}(x;u)| \|\tau^k\|_X \leq \frac{2\sqrt{2}(k+2\nu)}{C\sqrt{(2\nu)!}(2\nu+1)(k-1+2\nu)^{2\nu}} (\frac{r}{R})^{k-1} E_R^{k-1}(u) \leq \|u(\tau x)\|_X,$$

where C is a constant independent of u and τx.

Proof. Using the addition theorem [1] for the Gegenbauer polynomials C_k^ν of degree k and order ν, we have

$$\int_{S^n} C_k^\nu[(x,\zeta)] P(\tau\zeta) dS(\zeta) = 0$$

where $P \in \Pi_{k-1}, 0 < \tau < R, x \in S^n, \zeta \in S^n$, and

$$Y^{(k)}(x;u)r^k = \frac{2(k+\nu)}{d_n w_n} \int_{S^n} C_k^\nu[(x,y)] u(ry) dS(y), \tag{2.9}$$

where $k \in Z, d_2 = 1, d_n = n-2$ at $n > 2, \nu = \frac{n-2}{2}, w_n = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.
Rewrite (2.9) as

$$Y^{(k)}(x;u)\tau^k = \frac{(k+\nu)}{\nu w_n}\int_{S^n} C_k^\nu[(x,\zeta)][u(\tau\zeta) - P(\tau\zeta)]dS(\zeta). \tag{2.10}$$

Since $\max_{-1\le t\le 1}|C_k^\nu(t)| = C_k^\nu(1)$, from [1] we have $C_k^\nu(1) = \frac{(k+2\nu-1)!}{(d_n-1)!k!}$ and from (2.10) we obtain

$$\begin{aligned}|Y^{(k)}(x;u)|\|\tau^k\|_X \le &\frac{(k+\nu)}{\nu w_n}\|u(\tau\zeta) - P(\tau\zeta)\|_X C_k^\nu(1)w_n \\ \le &\frac{2(k+2\nu)^{2\nu}}{(2\nu)!}\|u(\tau\zeta) - P(\tau\zeta)\|_X\end{aligned}$$

there exists a polynomial $P^* \in \Pi_{k-1}$ for which

$$\|u(\tau\zeta) - P(\tau\zeta)\|_X \le C\max|u(\tau\zeta) - P^*(\tau\zeta)| \le 2E_R^{k-1}(u).$$

So we have

$$|Y^{(k)}(x;u)|\|\tau^k\|_X \le \frac{4C(k+2\nu)^{2\nu}}{(2\nu)!}E_R^{k-1}(u). \tag{2.11}$$

Now consider $Q(\tau\zeta) = \sum_{j=0}^k Y^{(j)}(\zeta;u)\tau^j$, since $Q \in \Pi_k$, we have

$$E_R^k(u) \le \|u(\tau\zeta) - Q(\tau\zeta)\|_X \le C\max_{\tau\zeta\in\overline{B_R^n}}|u(\tau\zeta) - Q(\tau\zeta)|,$$

Using a result of [27] we have

$$\begin{aligned}E_R^k(u) \le &\sum_{j=k+1}^{\infty} C\max_{\zeta\in S^n}|Y^{(j)}(\zeta,u)|R^j \\ \le &C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X\sum_{j=k+1}^{\infty}(j+2\nu)^\nu(\frac{R}{r})^j \\ = &C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k\sum_{j=k+1}^{\infty}(j+2\nu)^\nu(\frac{R}{r})^{j-k}.\end{aligned} \tag{2.12}$$

For $r > eR$, the maximum value of last sum can be estimate as

$$\sum_{j=k+1}^{\infty}(j+2\nu)^\nu(\frac{R}{r})^{j-k} \le e^k\sum_{j=k+1}^{\infty}(j+2\nu)^\nu e^{-j} \le e^k\int_k^\infty(t+2\nu)^{2\nu}e^{-t}dt.$$

Set $\eta = 2\nu, \theta_\eta(t) = (t+\eta)^\eta$ and integrating $(\eta+1)$ times by parts, we get

$$\int_k^\infty \theta_\eta(t)e^{-t}dt = [-e^{-t}(\theta_\eta(t) + \theta'_\eta(t) + \cdots + \theta_\eta^{(n)}(t))]|_k^\infty$$
$$= e^{-k}\sum_{i=0}^{\eta}\frac{\eta!(k+\eta)^{\eta-i}}{(\eta-i)!}, \quad since \quad \theta_\eta^{(i)} = \frac{\eta!}{(\eta-i)}(t+\eta)^{\eta-i}, i = \overline{1,\eta},$$
$$= e^{-k}\sum_{i=0}^{2\nu}\frac{(2\nu)!(k+2\nu)^{2\nu-1}}{(2\nu-i)!}.$$

The maximum value of above term is $(2\nu+1)!(k+2\nu)^{2\nu}$. Hence from (2.12) we have

$$E_R^k(u) \le C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k(2\nu+1)(k+2\nu)^{2\nu}. \tag{2.13}$$

Combining (2.11) with (2.13) we get

$$|Y^{(k)}(x;u)|\|\tau^k\|_X \le \frac{4(k+2\nu)^{2\nu}}{(2\nu)!}E_R^{k-1}(u) \le C\sqrt{\frac{2}{(2\nu)!}}\|u(\tau\zeta)\|_X(\frac{R}{r})^k(2\nu+1)(k+2\nu)^{2\nu}, \tag{2.14}$$

above inequality (2.14) gives the required result. □

Lemma 2.3. *Let*

$$\alpha_1 = \liminf_{k\to\infty}(\|\tau^k\|)^{\frac{1}{k}} \quad and \quad \alpha_2 = \limsup_{k\to\infty}(\|\tau^k\|)^{\frac{1}{k}}.$$

Then $\alpha_1 \ge R$ and $\alpha_2 < \infty$.

Proof. Suppose $\beta_k = (\|\tau^k\|)^{\frac{1}{k}}$. First we prove that $\alpha_2 < \infty$. On the contrary we assume that there exists a subsequence β_{k_m} such that $\lim_{m\to\infty}\beta_{k_m} = \infty$. Consider a function u_0 such that

$$u_0(\tau x) = \sum_{m=0}^{\infty}(\beta_{k_m})^{-\frac{k_m}{2}}\tau^{k_m}.$$

The function $u_0(\tau x)$ is entire and therefore belong to X. However, in this case using Lemma 2.2, for any $m \in \mathbb{N}$, we get

$$(\beta_{k_m})^{-\frac{k_m}{2}}\|\tau^{k_m}\| \le \|u_0(\tau x)\| < \infty,$$

which is impossible. Hence $\alpha_2 < \infty$. Now in order to prove $\alpha_1 \ge R$ we assume that $\alpha_1 < R$. Set $\delta \in (\alpha_1; R)$ and consider a function

$$u_0(\tau x) = \sum_{m=0}^{\infty}\delta^{-k_m}\tau^{k_m} \tag{2.15}$$

where k_m is a sequence such that $\liminf_{k\to\infty} \beta_k = \lim_{m\to\infty} \beta_{k_m} = \alpha_1$. The function $u_0(\tau x)$ is harmonic in the ball $B^n_\delta(u_0), \delta < R$ but not harmonic in ball B^n_R. It is clear that a sequence of partial sums $S_{k,u_0}(\tau x)$ of series (2.15) is fundamental in the Banach space X and, therefore, convergence in it to a function $u_1(\tau x) \in X$. We now prove that the Fourier-Laplace coefficients of the functions $u_0(\tau x)$ and $u_1(\tau x)$ are same. For fixed $m \in \mathbb{N} \cup \{0\}, k > m$, we have

$$Y^{(k)}(x; u_1) = Y^{(k)}(x; S_{k,u_0}) + Y^{(k)}(x; u_1 - S_{k,u_0}) = Y^{(k)}(x; u_0) + Y^{(k)}(u_1 - S_{k,u_0}).$$

Proceeding to the limit as $k \to \infty$ and using Lemma 2.2, it gives $Y^{(k)}(x; u_1) = Y^{(k)}(x; u_0)$. Hence the function $u_1(\tau x) \in X$ but not harmonic in B^n_R, which contradicts the property of the space X. Hence $\alpha_1 \geq R$. $\square$

Lemma 2.4. *Let $u \in X$ and let K be a compact subset of $\mathbb{R}^n, n \geq 3, K \subset B^n_R$. Then, for $\tau x \in K$,*

$$|u(\tau x)| \leq C\|u(\tau x)\|.$$

Proof. Let $\gamma = \sup\{|\tau x| : \tau x \in K\}, \gamma < R$. Now write the expansion of u in the Fourier-Laplace series and estimates its modulus by using Lemma 2.2, we get

$$u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)\tau^k,$$

$$|u(\tau x)| \leq \sum_{k=0}^{\infty} |Y^{(k)}(x; u)||\tau^k| \leq \|u(\tau x)\| \sum_{k=0}^{\infty} \frac{\gamma^k}{\|\tau^k\|} \leq C\|u(\tau x)\|$$

as the series is convergent by Lemma 2.3. $\square$

3. Main Results

Theorem 3.1. *Let $u \in X$. The condition*

$$\lim_{k\to\infty} (E^k_R(u))^{\frac{1}{k}} = 0$$

is necessary and sufficient for the function u to be entire.

Proof. Let $u(\tau x) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)\tau^k$ in space $\mathbb{R}^n, n \geq 3, 0 < \tau < R$. In view of Lemma 2.2,

$$|Y^{(k)}(x; u)|\|\tau^k\|_X \leq \frac{2\sqrt{2}(k + 2\nu)}{C\sqrt{(2\nu)!}(2\nu + 1)(k - 1 + 2\nu)^{2\nu}} (\frac{r}{R})^{k-1} E^{k-1}_R(u)$$

it gives

$$\lim_{k\to\infty} |Y^{(k)}(x; u)|^{\frac{1}{k}} \leq \lim_{k\to\infty} (\frac{2\sqrt{2}(k + 2\nu)}{C\sqrt{(2\nu)!}(2\nu + 1)(k - 1 + 2\nu)^{2\nu}} (\frac{r}{R})^{k-1} \frac{E^{k-1}_R(u)}{\|\tau^k\|_X})^{\frac{1}{k}} = 0,$$

therefore the function u is entire. Now for necessity, from Lemma 2.3, we have

$$\frac{E_R^{k-1}(u)R^k}{\|\tau^k\|_X} \le \frac{\sqrt{(2\nu)!}(2\nu+1)(k-1+2\nu)\|u(\tau x)\|_X R^k r}{2\sqrt{2}(k+2\nu)\|\tau^k\|_X r^k}.$$

Since u is entire and $u \in X$ for any $r > 1$, we have

$$0 \le \lim_{k\to\infty}(\frac{E_R^{k-1}(u)R^k}{\|\tau^k\|_X})^{\frac{1}{k}} \le \frac{1}{r}\limsup_{k\to\infty}(\frac{\sqrt{(2\nu)!}(2\nu+1)(k-1+2\nu)\|u(\tau x)\|_X R^k r}{2\sqrt{2}(k+2\nu)\|\tau^k\|_X})^{\frac{1}{k}} \le \frac{1}{r}.$$

Now by arbitrariness of $r > 1$, we obtain

$$\lim_{k\to\infty}(E_R^k(u))^{\frac{1}{k}} = 0.$$

The proof of Theorem 3.1 is completed. $\square$

Theorem 3.2. *For a function $u \in X$ to be an entire harmonic function in space $\mathbb{R}^n, n \ge 3$, of finite order $0 < \rho < \infty$, the necessary and sufficient condition is*

$$\limsup_{k\to\infty}\frac{k\log k}{\log(\frac{\|\tau^k\|}{E_R^k(u)})} = \rho. \tag{3.1}$$

Proof. To prove sufficiency part let (3.1) holds therefore the condition of Theorem 3.1 is satisfied and, hence, the function u is entire harmonic in space $\mathbb{R}^n, n \ge 3$ and we denote its order by ρ_1. Thus, on account of Lemma 2.2 we obtain

$$\rho_1 = \limsup_{k\to\infty}\frac{k\log k}{\log|Y^{(k)}(x;u)|} \le \limsup_{k\to\infty}\frac{k\log k}{\log(\frac{\|\tau^k\|}{E_R^k(u)})} = \rho. \tag{3.2}$$

According to the condition of theorem we have to show that $\rho_1 > 0$. On the contrary, we assume that

$$\limsup_{k\to\infty}\frac{k\log k}{\log|Y^{(k)}(x;u)|} = 0.$$

Then for any $\varepsilon, 0 < \varepsilon < R$, there exists K_ε such that, for $k > K_\varepsilon$

$$k\log k < -\varepsilon\log|Y^{(k)}(x;u)|$$

or

$$|Y^{(k)}(x;u)| < k^{-\frac{k}{\varepsilon}}.$$

Now for sufficiently large K_ε we have

$$\|\tau^k\|_X \le (\mu_2+\varepsilon)^k \quad \text{and} \quad \|\tau^k\|_X \ge (R-\varepsilon)^k \quad \text{for} \quad k \ge K_\varepsilon.$$

Then

$$E_R^k(u) \leq \| \sum_{j=k+1}^{\infty} |Y^{(k)}(x;u)|\tau^k\|_X \leq \sum_{j=k+1}^{\infty} j^{-\frac{j}{\varepsilon}}(\mu_2+\varepsilon)^j$$
$$\leq \sum_{j=k+1}^{\infty}(k+1)^{-\frac{j}{\varepsilon}}(\mu_2+\varepsilon)^j = (k+1)^{-\frac{(k+1)}{\varepsilon}}(\mu_2+\varepsilon)^{k+1}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\varepsilon}}})^{-1}. \tag{3.3}$$

We assume $k+1 \geq (\mu_2+\varepsilon)^{\varepsilon}$ in (3.3), it gives

$$\frac{\|\tau^k\|_X}{E_R^k(u)} \geq (\frac{R-\varepsilon}{\mu_2+\varepsilon})^{k+1}(k+1)^{\frac{(k+1)}{\varepsilon}}(1-\frac{\mu_2+\varepsilon}{(k+1)^{\frac{1}{\varepsilon}}}),$$

or

$$\log(\frac{\|\tau^k\|_X}{E_R^k(u)})^{\frac{1}{k}} \geq (\frac{k+1}{k})\log(\frac{R-\varepsilon}{\mu_2+\varepsilon}) + \frac{k+1}{k\varepsilon}\log(k+1) + \frac{1}{k}\log(1-\frac{\mu_2+\varepsilon}{(k+1)^{\frac{1}{\varepsilon}}})$$

or

$$\liminf_{k\to\infty}\frac{\log(\frac{\|\tau^k\|_X}{E_R^k(u)})^{\frac{1}{k}}}{\log k} \geq \frac{1}{\varepsilon},$$

or

$$\rho = \limsup_{k\to\infty}\frac{k\log k}{\log\frac{\|\tau^k\|_X}{E_R^k(u)}} \leq \varepsilon,$$

which contradicts our assumption. Now we consider the case for $\varepsilon \in (0,\frac{R}{2}) \cap (0,\rho_1)$. From the left hand side of (3.2) we conclude that there exists $K_\varepsilon \in \mathbb{N}(\varepsilon)$ such that

$$|Y^{(k)}(x;u)| < k^{-\frac{k}{(\rho_1+\varepsilon)}}$$

for all $k > K_\varepsilon$. Let K_ε be sufficiently large such that $\|\tau^k\|_X \leq (\mu_2+\varepsilon)$ and $\|\tau^k\| \geq (R-\varepsilon)^k$ for $k \geq K_\varepsilon$. Then for $k > K_\varepsilon$,

$$E_R^k(u) \leq \|\sum_{j=k+1}^{\infty} Y^{(j)}(x;u)\tau^j\| \leq \sum_{j=k+1}^{\infty}|Y^{(j)}(x;u)|\|\tau^k\|$$
$$\leq \sum_{j=k+1}^{\infty}(j)^{-\frac{j}{\rho_1+\varepsilon}}\|\tau^j\| \leq \sum_{j=k+1}^{\infty}(k+1)^{-\frac{j}{\rho_1+\varepsilon}}(\mu_2+\varepsilon)^j \tag{3.4}$$
$$= \frac{(\mu_2+\varepsilon)^{k+1}}{(k+1)^{\frac{(k+1)}{(\rho_1+\varepsilon)}}}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}})^{-1},$$

or

$$\frac{\|\tau^k\|(k+1)^{\frac{1}{\rho_1+\varepsilon}}}{E_R^k(u)} \geq \frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{k+1}}(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}}),$$

or

$$\rho_1+\varepsilon \geq \frac{(k+1)\log(k+1)}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}}(1+\frac{(\rho_1+\varepsilon)}{(k+1)\log(k+1)}\log(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho_1+\varepsilon}}})+ \frac{\rho_1+\varepsilon}{(k+1)\log(k+1)}\log\frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{(k+1)}}). \tag{3.5}$$

Proceeding to limit as $k \to \infty$, we get $\rho_1+\varepsilon \geq \rho$. Since ε is arbitrary this implies that $\rho_1 \geq \rho$. In view of (3.2), we get $\rho_1 = \rho$, hence the sufficient part is completed. In order to prove the necessary part we assume that $u \in X$ be an entire harmonic function of finite order ρ, i.e.,

$$\limsup_{k\to\infty} \frac{k\log k}{-\log|Y^{(k)}(x;u)|} = \rho.$$

Set

$$\rho_1 = \limsup_{k\to\infty} \frac{k\log k}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}}.$$

Here ρ_1 and ρ are interchanged as compared with the proof of sufficiency part and show that $\rho_1 = \rho$. By analogy with (3.2), Lemma 2.2 gives $\rho_1 \geq \rho$. Following the same fact as in the sufficiency part, we can say that, for any $\varepsilon, 0<\varepsilon<R$ there exists K_ε such that

$$|Y^{(k)}(x;u)| < k^{-\frac{k}{(\rho+\varepsilon)}} \quad \textit{and} \quad (R-\varepsilon)^k \leq \|\tau^k\| \leq (\mu_2+\varepsilon)^k$$

for $k > K_\varepsilon$. Following (3.3) and (3.4) (with ρ_1 and ρ interchanged), we get

$$\rho+\varepsilon \geq \frac{(k+1)\log(k+1)}{\log \frac{\|\tau^k\|_X}{E_R^k(u)}}(1+\frac{(\rho+\varepsilon)}{(k+1)\log(k+1)}\log(1-\frac{(\mu_2+\varepsilon)}{(k+1)^{\frac{1}{\rho+\varepsilon}}})+ \frac{\rho+\varepsilon}{(k+1)\log(k+1)}\log\frac{\|\tau^k\|}{(\mu_2+\varepsilon)^{(k+1)}}).$$

Proceeding to limit as $k \to \infty$, it gives $\rho \geq \rho_1$. Hence the proof is completed. □

Theorem 3.3. For a function $u \in X$ to be an entire harmonic function of finite order $\rho \in (0,\infty)$ and normal type $\sigma \in (0,\infty)$, the necessary and sufficient condition is that

$$\limsup_{k\to\infty} \frac{k}{e\rho}(\frac{E_R^k(u)}{\|\tau^k\|})^{\frac{\rho}{k}} = \sigma. \tag{3.6}$$

Proof. In order to prove sufficiency we assume that $u \in X$ and satisfies the condition of Theorem 3.3 with some positive ρ and σ. Then (3.1) follows from (3.6), therefore, u is an entire harmonic function of order ρ. Assume that the type of u is T. We have to prove $T=\sigma$. Now using the classical coefficient formula for the type of an entire harmonic function $u\in X$

$$T=\limsup_{k\to\infty}\frac{k}{e\rho}|Y^{(k)}(x;u)|^{\frac{\rho}{k}} \tag{3.7}$$

with Lemma 2.3, we obtain $T\leq\sigma$. To prove the reverse inequality we have from (3.7) that for any $\varepsilon>0$, there exists $K_\varepsilon\in\mathbb{N}$ such that, for $k>K_\varepsilon$,

$$|Y^{(k)}(x;u)|<(\frac{\rho e(T+\varepsilon)}{k})^{\frac{k}{\rho}}. \tag{3.8}$$

Following the same techniques as (3.4) and (3.5), we have from (3.8),

$$E_R^k(u)\leq\sum_{j=k+1}^{\infty}(\frac{\rho e(T+\varepsilon)}{j})^{\frac{j}{\rho}}\|\tau^j\|\leq(\frac{\rho e(T+\varepsilon)}{k+1})^{\frac{k+1}{\rho}}(\mu+\varepsilon)^{(k+1)}(1-\frac{C^*}{(k+1)^{\frac{1}{\rho}}})^{-1} \tag{3.9}$$

where $C^*=(\mu+\varepsilon)(\rho e(T+\varepsilon))^{\frac{1}{\rho}}$. Now in view of (3.9), we obtain

$$T+\varepsilon\geq\frac{(k+1)}{e\rho}(\frac{E_R^k(u)}{\|\tau^k\|})^{\frac{\rho}{(k+1)}}\frac{\|\tau^k\|^{\frac{\rho}{(k+1)}}}{(\mu+\varepsilon)^\rho}(1-\frac{C^*}{(k+1)^{\frac{1}{\rho}}})^{\frac{\rho}{(k+1)}}.$$

Proceeding the limit sup as $k\to\infty$, we get

$$T+\varepsilon\geq\sigma(\frac{\mu}{\mu+\varepsilon})^\rho.$$

Since ε is arbitrary and approaches to zero, we get $T\geq\sigma$. Hence the sufficiency part is completed.
Now to prove necessity assume that $u\in X$ is an entire harmonic function of finite order and normal type. We denote its order and type as ρ and T respectively. Further, we have to show that $T=\sigma$. By virtue of (3.7) and Lemma 2.3, we obtain $T\leq\sigma$. Finally, to prove $\sigma\leq T$ repeat the reasoning of sufficiency part. This completes the proof of necessary part. Hence the proof of theorem is completed.

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Weakly Locally Uniformly Rotund Norm which is not Locally Uniformly Rotund

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ABSTRACT: The aim of this paper is to provide a proof of the fact that a weakly locally uniformly rotund norm does not have to be locally uniformly rotund. This result is well-known for experts in Geometry of Banach Spaces. However, since the justification of this result is omitted in the literature, we believe that the present note may be helpful for students or novices in the theory.

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1. Introduction

The notion of locally uniformly rotund space was introduced by A. R. Lovaglia in [3] and widely studied afterwards. In our considerations we will accept the following definition of the concept of the (weak) local uniform rotundity (cf. [1]).

Definition 1.1. A normed space $(X, \|\cdot\|)$ is *locally uniformly rotund* (*weakly locally uniformly rotund*) if for $x \in X$, $(x_n) \subset X$, such that $\|x\| = 1$, $\|x_n\| = 1$ for $n \in \mathbb{N}$ and

$$\left\| \frac{x_n + x}{2} \right\| \to 1,$$

we get that x_n converges to x (x_n converges weakly to x).

Let us consider the space c_0 of the sequences convergent to zero with the norm given by the formula

$$|||x||| = \|x\|_\infty + \left(\sum_{k=1}^{\infty} 2^{-k}|x_k|^2\right)^{\frac{1}{2}} \tag{1.1}$$

for $x = (x_k) \in c_0$.

In [4] authors used c_0 with the norm (1.1) as an example of rotund norm, which is not locally uniformly rotund. Later, S. Draga proved in [2] that the considered norm is weakly locally uniformly rotund. It is worthwhile mentioning that there is no proof of the fact observed in [4] and to our best knowledge there is no such proof in other papers (a similar example in the space $C[0,1]$ together with a proof was established in [3], pp. 229-230). In this paper we fill that gap. Namely, we will show now that the norm (1.1) on the space c_0 is not locally uniformly rotund.

2. The proof

Let $x = \left(2-\sqrt{2}, 0, 0, ...\right) \in c_0$ ($|||x||| = 1$). Consider $(x_n) \subset c_0$, where

$$x_n = \left(\alpha_n, 0, 0, \ldots, 0, \frac{1}{2}, 0, 0, \ldots\right)$$

and $\frac{1}{2}$ lies on $n+1$ coordinate, $\alpha_n > 0$. We want to choose α_n such that $|||x_n||| = 1$ for $n \in \mathbb{N}$. We are going to explain how to fulfill this requirement. We have

$$1 = |||x_n||| = \alpha_n + \sqrt{\frac{1}{2}\alpha_n^2 + \left(\frac{1}{2}\right)^{n+1} \cdot \left(\frac{1}{2}\right)^2},$$

$$\frac{1}{2}\alpha_n^2 - 2\alpha_n + 1 - \left(\frac{1}{2}\right)^{n+3} = 0.$$

Hence, we obtain

$$\alpha_n = 2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} \quad \text{ or } \quad \alpha_n = 2 + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}.$$

Since $1 - \alpha_n \geq 0$, we conclude that

$$x_n = \left(2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}, 0, 0, \ldots, 0, \frac{1}{2}, 0, 0, \ldots\right).$$

Hence, we get

$$\left|\left|\left|\frac{x_n + x}{2}\right|\right|\right| = \left|\left|\left|\left(\frac{2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} + 2 - \sqrt{2}}{2}, 0, 0, \ldots, 0, \frac{1}{4}, 0, 0, \ldots\right)\right|\right|\right|$$

$$= \frac{4 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} - \sqrt{2}}{2} + \sqrt{\frac{1}{2}\left(\frac{4 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} - \sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^{n+1}\left(\frac{1}{4}\right)^2}$$

$$= 2 - \frac{\sqrt{2} + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}}{2} + \sqrt{\frac{1}{2}\left(2 - \frac{\sqrt{2} + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}}{2}\right)^2 + \left(\frac{1}{2}\right)^{n+5}}.$$

Passing with n to infinity, we obtain

$$\left|\!\left|\!\left|\frac{x_n + x}{2}\right|\!\right|\!\right| \xrightarrow[n\to\infty]{} 2 - \sqrt{2} + \sqrt{\frac{1}{2}}\left(2 - \sqrt{2}\right) = 1.$$

On the other hand, we have

$$|\!|\!|x_n - x|\!|\!| \geq \|x_n - x\|_\infty = \left\|\left(\sqrt{2} - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}, 0, 0, \dots, 0, \frac{1}{2}, 0, 0, \dots\right)\right\|_\infty \geq \frac{1}{2},$$

for all $n \in \mathbb{N}$. This shows that the sequence x_n does not converge to x in the norm $|\!|\!|\cdot|\!|\!|$. As a consequence, the norm $|\!|\!|\cdot|\!|\!|$ defined by (1.1) is not locally uniformly rotund. $\square$

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Zero-sum Games on a Product of Staircase-Function Finite Spaces

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ABSTRACT: A tractable method of solving zero-sum games defined on a product of staircase-function finite spaces is presented. The method is based on stacking solutions of "smaller" matrix games, each defined on an interval where the pure strategy value is constant. The stack is always possible, even when only time is discrete, so the set of pure strategy possible values can be continuous. Any combination of the solutions of the "smaller" matrix games is a solution of the initial zero-sum game..

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1. Introduction

Zero-sum games are pretty easy-to-understand models of processes where two sides (personified with a purpose to be referred to as persons or players) interact in struggling for optimizing the to-be-paid-or-pay situations [14, 15, 8, 10]. Matrix games are the simplest zero-sum games whose optimal solutions are well-studied [15, 8]. Antagonistic games (where the game kernel is a surface, which may have also discontinuities, defined on a finite-dimensional Euclidean subspace) are more complicated as, opposed to matrix games, the optimal solution is not always determinable and feasible [8, 7, 15]. Therefore, the best choice is to approximate such games to finite ones, which are easily rendered to matrix games [8]. Nevertheless, even a matrix game solution, if it is in mixed strategies, is not always practicable due to finite horizon of the game iterations (actions, plays, etc.) [9, 10, 11, 12].

A far more complicated case is a zero-sum game, in which the player's (pure) strategy is a function (e. g., of time). In such games, the payoff kernel is a functional

[13]. This functional maps every pair of functions (pure strategies of the players) into a real value. In the case, when each of the players possesses a finite set of such function-strategies, the game might be rendered down to a matrix game [8, 10, 12]. Such rendering is impossible if the set of the player's function-strategies is either infinite or continuous.

If to break a time interval, on which the pure strategy is defined, into a set of subintervals, on which the strategy could be approximately considered constant, the game is not simplified much. The matter is the continuity of possible values of the strategy on a subinterval. The continuity might be removed also by discretization (or sampling) [12, 13, 7]. Thus, the set of function-strategies becomes finite. Therefore, an appropriate transformation of the continuous spaces of function-strategies into finite spaces is a problem of high priority.

2. Motivation

The case when the set of the player's pure strategies is finite seems to be too plain and not reflecting true practical problems. However, the number of factual actions of the players (in any game) is always finite. While the players may use strategies of whichever form they want, the eventual number of their actions has a natural limit [1, 10, 2, 4, 5]. Therefore, if the rules of a system to be game-modeled are defined and administered beforehand, the administrator is likely to define (or constrain) the form of the strategies players will use [16, 17].

In the simplest case, the player's pure strategy is a simple action whose duration is negligibly short and can be represented as just a time point. This case is exhaustively studied as matrix games [15, 8, 12]. In a more complicated case, the player's pure strategy is a function of time [13]. The time interval is usually short, although its length does not matter much [2, 10, 1]. A way to appropriately administer the players' actions is to constrain them to staircase functions whose points of discontinuities (breakpoints) have to be the same for both the players [5, 17, 3]. Along with the discrete time, possible values of the player's pure strategy should be discrete as well. Then, due to the set of the player's possible actions is finite, the game can be represented as a matrix game, in which the player's selection of a pure strategy means using a staircase (time) function on an interval whereon every pure strategy is defined. Obviously, the number of the player's pure strategies in the matrix staircase-function game grows immensely as the number of breakpoints ("stair" intervals) or/and the number of possible values of the player's pure strategy increases. For instance, if the number of intervals is 10, and the number of possible values of the player's pure strategy is 6, then there are

$$6^{10} = 60466176$$

possible pure strategies (10-interval staircase functions of time) at this player. The respective matrix staircase-function game appears to be intractably gigantic: if even the other player has just two possible values of one's pure strategy, then there are

$$2^{10} = 1024$$

possible pure strategies, and the resulting payoff matrix is of

$$6^{10} \cdot 2^{10} = 61917364224$$

entries (more than 61.9 billion entries!). Therefore, a tractable method of solving zero-sum games defined on a product of staircase-function finite spaces should be suggested.

3. Goal and tasks to be fulfilled

Due to the above reasons, the goal is to develop a tractable method of solving zero-sum games defined on a product of staircase-function finite spaces. For achieving the goal, the following tasks are to be fulfilled:

1. To formalize a zero-sum game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Herein, the time can be thought of as it is discrete.

2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.

3. To formalize a method of solving zero-sum games defined on a product of staircase-function finite spaces.

4. To discuss applicability and significance of the method for the game theory, whereupon an unbiased conclusion is to be made.

4. A zero-sum game with staircase-function strategies

A zero-sum game, in which the player's pure strategy is a function of time, can be defined as follows. Let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the first player by $x(t)$ and a strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}, \tag{2}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2] \tag{3}$$

and

$$Y = \{y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset$$

$$\subset \mathbb{L}_2 \left[t_1;\ t_2\right] \tag{4}$$

are the sets of the players' pure strategies.

The first player's payoff in situation $\{x(t),\ y(t)\}$ is $K(x(t),\ y(t))$. The payoff is presumed to be an integral functional:

$$K(x(t),\ y(t)) = \int\limits_{[t_1;\ t_2]} f(x(t),\ y(t),\ t)\, d\mu(t), \tag{5}$$

where $f(x(t),\ y(t),\ t)$ is a function of $x(t)$ and $y(t)$ explicitly including t. Therefore, the continuous zero-sum (two-person) game

$$\langle \{X,\ Y\},\ K(x(t),\ y(t)) \rangle \tag{6}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2 \left[t_1;\ t_2\right] \times \mathbb{L}_2 \left[t_1;\ t_2\right] \tag{7}$$

of rectangular functional spaces (3) and (4) of players' pure strategies.

Presume that the players' pure strategies $x(t)$ and $y(t)$ in game (6) can be mapped to functions which both change their values for a finite number of times so that the optimal strategies in the new game are equivalent (i. e., are different only on a set of the null measure) to the optimal strategies in game (6) by (7). Denote by N the number of intervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only N different values. If $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2, \tag{8}$$

then

$$x_i = x\left(\tau^{(i)}\right) \ \text{ by } \ i = \overline{0,\ N} \tag{9}$$

are the values of the first player's strategy, and

$$y_i = y\left(\tau^{(i)}\right) \ \text{ by } \ i = \overline{0,\ N} \tag{10}$$

are the values of the second player's strategy. The staircase-function strategies are right-continuous:

$$\lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \tag{11}$$

and

$$\lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} y\left(\tau^{(i)} + \varepsilon\right) = y\left(\tau^{(i)}\right) \tag{12}$$

for $i = \overline{1,\ N-1}$, whereas

$$\lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} x\left(\tau^{(i)} - \varepsilon\right) \neq x\left(\tau^{(i)}\right) \tag{13}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right) \neq y\left(\tau^{(i)}\right) \tag{14}$$

for $i=\overline{1,\ N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right) = x\left(\tau^{(N)}\right) \tag{15}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right) = y\left(\tau^{(N)}\right), \tag{16}$$

so $x_{N-1}=x_N$ and $y_{N-1}=y_N$. Then constant values (9) and (10) by (8) mean that game (6) can be thought of as it is a succession of N continuous games

$$\left\langle \left\{ \left[a_{\min};\ a_{\max}\right],\ \left[b_{\min};\ b_{\max}\right] \right\},\ K\left(\alpha_i,\ \beta_i\right) \right\rangle \tag{17}$$

defined on product

$$\left[a_{\min};\ a_{\max}\right] \times \left[b_{\min};\ b_{\max}\right] \tag{18}$$

by

$$\begin{gathered} \alpha_i = x\left(t\right) \in \left[a_{\min};\ a_{\max}\right] \text{ and } \beta_i = y\left(t\right) \in \left[b_{\min};\ b_{\max}\right] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i=\overline{1,\ N-1} \text{ and } \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right], \end{gathered} \tag{19}$$

where the factual payoff in situation $\{\alpha_i,\ \beta_i\}$ is

$$K\left(\alpha_i,\ \beta_i\right) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ t\right) d\mu\left(t\right) \ \forall\, i=\overline{1,\ N-1} \tag{20}$$

and

$$K\left(\alpha_N,\ \beta_N\right) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(\alpha_N,\ \beta_N,\ t\right) d\mu\left(t\right), \tag{21}$$

so

$$\begin{gathered} K\left(x\left(t\right),\ y\left(t\right)\right) = \\ = \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ t\right) d\mu\left(t\right) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(\alpha_N,\ \beta_N,\ t\right) d\mu\left(t\right) \end{gathered} \tag{22}$$

instead of (5). In other words, if every optimal situation in pure strategies in game (6) on product (7) by conditions (1) — (5) is of staircase functions satisfying conditions (8) — (16), then this game is equivalent to the succession of N games (17) by (8) — (16) and (19). In this case game (6) can be represented by the succession of games (17). On the other hand, the equivalency can be settled after a requirement of that the players in game (6) use only their pure strategies in the form of staircase functions.

Theorem 1. *If each of N games (17) by (8) — (16) and (19) has a single optimal situation in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then the optimal situation in pure strategies in game (6) is single and can be determined by independently solving N games (17).*

Proof. First, the equivalency means that game (6) has only staircase solutions. Next, it should be proved that game (6) has a single optimal situation in pure strategies. Let $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^{N}$ be optimal solutions in games (17) by (8) — (16) and (19). Then

$$\max_{\alpha_i \in [a_{\min};\ a_{\max}]} \ \min_{\beta_i \in [b_{\min};\ b_{\max}]} K(\alpha_i,\ \beta_i) =$$
$$= \max_{\alpha_i \in [a_{\min};\ a_{\max}]} \ \min_{\beta_i \in [b_{\min};\ b_{\max}]} \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) =$$
$$= \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i^*,\ \beta_i^*,\ t)\, d\mu(t) =$$
$$= \min_{\beta_i \in [b_{\min};\ b_{\max}]} \ \max_{\alpha_i \in [a_{\min};\ a_{\max}]} \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) =$$
$$= \min_{\beta_i \in [b_{\min};\ b_{\max}]} \ \max_{\alpha_i \in [a_{\min};\ a_{\max}]} K(\alpha_i,\ \beta_i) \quad \forall\, i = \overline{1,\ N-1} \tag{23}$$

and

$$\max_{\alpha_N \in [a_{\min};\ a_{\max}]} \ \min_{\beta_N \in [b_{\min};\ b_{\max}]} K(\alpha_N,\ \beta_N) =$$
$$= \max_{\alpha_N \in [a_{\min};\ a_{\max}]} \ \min_{\beta_N \in [b_{\min};\ b_{\max}]} \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) =$$
$$= \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N^*,\ \beta_N^*,\ t)\, d\mu(t) =$$
$$= \min_{\beta_N \in [b_{\min};\ b_{\max}]} \ \max_{\alpha_N \in [a_{\min};\ a_{\max}]} \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) =$$
$$= \min_{\beta_N \in [b_{\min};\ b_{\max}]} \ \max_{\alpha_N \in [a_{\min};\ a_{\max}]} K(\alpha_N,\ \beta_N)\,. \tag{24}$$

So,

$$\max_{x(t) \in X} \ \min_{y(t) \in Y} K(x(t)\,,\ y(t)) =$$
$$= \sum_{i=1}^{N-1} \left(\max_{\alpha_i \in [a_{\min};\ a_{\max}]} \ \min_{\beta_i \in [b_{\min};\ b_{\max}]} \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \right) +$$

$$+\max_{\alpha_N\in[a_{\min};\,a_{\max}]}\ \min_{\beta_N\in[b_{\min};\,b_{\max}]}\ \int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(\alpha_N,\ \beta_N,\ t\right)d\mu\left(t\right)=$$

$$=\sum_{i=1}^{N-1}\ \int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(\alpha_i^*,\ \beta_i^*,\ t\right)d\mu\left(t\right)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(\alpha_N^*,\ \beta_N^*,\ t\right)d\mu\left(t\right)=$$

$$=\sum_{i=1}^{N-1}\left(\min_{\beta_i\in[b_{\min};\,b_{\max}]}\ \max_{\alpha_i\in[a_{\min};\,a_{\max}]}\ \int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ t\right)d\mu\left(t\right)\right)+$$

$$+\min_{\beta_N\in[b_{\min};\,b_{\max}]}\ \max_{\alpha_N\in[a_{\min};\,a_{\max}]}\ \int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(\alpha_N,\ \beta_N,\ t\right)d\mu\left(t\right)=$$

$$=\min_{y(t)\in Y}\ \max_{x(t)\in X} K\left(x\left(t\right),\ y\left(t\right)\right) \tag{25}$$

and, therefore, the stack of successive solutions $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^N$ is an optimal solution in game (6). Assume that there is another staircase solution in game (6). Then at least one game of those N games (17) exists having at least two different solutions in pure strategies. This is a contradiction, which means that game (6) has a single optimal situation in pure strategies, in which the player's optimal strategy is a stack of the respective optimal strategies in games (17). As every optimal strategy is single, games (17) can be solved independently, whose solutions are stacked afterwards to form the optimal solution in game (6). □

In fact, Theorem 1 claims that if the single pure-strategy solution of game (6) comprises staircase functions, and each of games (17) has a single pure-strategy solution, then the solution of game (6) can be determined in a simpler way, by solving games (17). They are solved in parallel, without caring of the succession. It is clear that an inverse assertion to Theorem 1 is correct as well. If it is only known that game (6) has a single optimal situation in pure strategies of staircase functions, then it does mean that the N constant parts of the player's optimal strategy can be directly taken as the solutions of respective games (17). Obviously, Theorem 1 can be generalized as follows.

Theorem 2. *If each of N games (17) by (8) — (16) and (19) has a nonempty set of optimal situations in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then every player's optimal strategy in pure strategies in game (6) is a combination of any respective N constant parts of the player's optimal strategy in games (17). Apart from the combination, there are no other optimal strategies in game (6).*

Proof. Let $\{\alpha_i^*,\ \beta_i^*\}_{i=1}^N$ be solutions in games (17) by (8) — (16) and (19). Then (23) — (25) hold, so the stack of these solutions is an optimal solution in game (6). First, consider the case, when each of games (17) has a finite set of optimal solutions.

Let the i-th game by (19) have n_i solutions, $i = \overline{1,\ N}$. Then the total number of the stacked situations is $\prod_{i=1}^{N} n_i$. Equilibria (23) and (24) hold for every such situation, so equilibrium (25) holds as well and thus game (6) has at least $\prod_{i=1}^{N} n_i$ optimal solutions. As the equivalency means that game (6) has only staircase solutions, assume that it has a solution which differs from those $\prod_{i=1}^{N} n_i$ situations. Then at least $\exists\, i_* \in \left\{\overline{1,\ N}\right\}$ such that the i_*-th game has at least $n_{i_*}+1$ different solutions in pure strategies. This is a contradiction, which means that game (6) has exactly $\prod_{i=1}^{N} n_i$ optimal situations in pure strategies. The cases of infinite and continuous sets of optimal situations in pure strategies in games (17) are considered and proved analogously. □

It is easy to see that, just like Theorem 1, Theorem 2 is invertible. The parts of the player's optimal strategy of respective games (17) can be taken from any optimal strategy of this player in game (6). Nevertheless, an exhaustive enumeration of all the solutions (in the case of only finite sets of optimal solutions) may be intractable as the number of games (17) or, that is the same, the number of intervals increases. For instance, if $N = 10$ and $n_i = 4 \quad \forall\, i = \overline{1,\ 10}$ then there are 1048576 solutions in game (6), whereupon it is quite hard to rank them or select the best one.

5. Representation by a series of matrix games

Along with discrete time intervals, players may be forced to act within a finite subset of possible values of their pure strategies. That is, these values are

$$a_{\min} = a^{(0)} < a^{(1)} < a^{(2)} < \ldots < a^{(M-1)} < a^{(M)} = a_{\max} \tag{26}$$

and

$$b_{\min} = b^{(0)} < b^{(1)} < b^{(2)} < \ldots < b^{(Q-1)} < b^{(Q)} = b_{\max} \tag{27}$$

for the first and second players, respectively ($M \in \mathbb{N}$ and $Q \in \mathbb{N}$). Then the succession of N continuous games (17) by (8) — (16) and (19) becomes a succession of N matrix games

$$\left\langle \left\{ \left\{ a^{(m-1)} \right\}_{m=1}^{M+1},\ \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1} \right\},\ \mathbf{K}_i \right\rangle \tag{28}$$

with payoff matrices $\mathbf{K}_i = [k_{imq}]_{(M+1)\times(Q+1)}$ whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \ \text{for} \ i = \overline{1,\ N-1} \tag{29}$$

and

$$k_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t). \tag{30}$$

So, if game (6) is made equivalent to a series of matrix games (or, in other words, is represented by a series of matrix games), then it is easy to see that, unlike the representation with continuous games (17) by (8) — (16) and (19), the game always has a solution (at least, in mixed strategies).

Theorem 3. *If game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of N matrix games (28) by (29) and (30), then the game is always solved as a stack of optimal solutions in these matrix games.*

Proof. A matrix game is always solved, either in pure or mixed strategies. Denote by

$$\mathbf{P}_{i\alpha} = \left[p_{i\alpha}^{(m)}\right]_{1\times(M+1)}$$

and

$$\mathbf{P}_{i\beta} = \left[p_{i\beta}^{(q)}\right]_{1\times(Q+1)}$$

the mixed strategies of the first and second players, respectively, in matrix game (28). The respective sets of mixed strategies of the first and second players are

$$\mathcal{P}_{\alpha} = \left\{\mathbf{P}_{i\alpha} \in \mathbb{R}^{M+1} : p_{i\alpha}^{(m)} \geqslant 0,\ \sum_{m=1}^{M+1} p_{i\alpha}^{(m)} = 1\right\}$$

and

$$\mathcal{P}_{\beta} = \left\{\mathbf{P}_{i\beta} \in \mathbb{R}^{Q+1} : p_{i\beta}^{(q)} \geqslant 0,\ \sum_{q=1}^{Q+1} p_{i\beta}^{(q)} = 1\right\},$$

so $\mathbf{P}_{i\alpha} \in \mathcal{P}_{\alpha}$, $\mathbf{P}_{i\beta} \in \mathcal{P}_{\beta}$, and $\{\mathbf{P}_{i\alpha},\ \mathbf{P}_{i\beta}\}$ is a situation in this game. Let

$$\left\{\mathbf{P}_{i\alpha}^{*},\ \mathbf{P}_{i\beta}^{*}\right\}_{i=1}^{N} = \left\{\left[p_{i\alpha}^{(m)*}\right]_{1\times(M+1)},\ \left[p_{i\beta}^{(q)*}\right]_{1\times(Q+1)}\right\}_{i=1}^{N}$$

be optimal solutions in N games (28) by (29) and (30). Then

$$\max_{\mathbf{P}_{i\alpha}\in\mathcal{P}_{\alpha}} \min_{\mathbf{P}_{i\beta}\in\mathcal{P}_{\beta}} \mathbf{P}_{i\alpha} \cdot \mathbf{K}_i \cdot \mathbf{P}_{i\beta}^{\mathrm{T}} =$$

$$= \max_{\mathbf{P}_{i\alpha}\in\mathcal{P}_{\alpha}} \min_{\mathbf{P}_{i\beta}\in\mathcal{P}_{\beta}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} =$$

$$= \max_{\mathbf{P}_{i\alpha}\in\mathcal{P}_{\alpha}} \min_{\mathbf{P}_{i\beta}\in\mathcal{P}_{\beta}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\,(t) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)*} p_{i\beta}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \mathbf{P}_{i\alpha}^{*} \cdot \mathbf{K}_{i} \cdot \left(\mathbf{P}_{i\beta}^{*}\right)^{\mathrm{T}} =$$

$$= \min_{\mathbf{P}_{i\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{i\alpha} \in \mathcal{P}_{\alpha}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \min_{\mathbf{P}_{i\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{i\alpha} \in \mathcal{P}_{\alpha}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} =$$

$$= \min_{\mathbf{P}_{i\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{i\alpha} \in \mathcal{P}_{\alpha}} \mathbf{P}_{i\alpha} \cdot \mathbf{K}_{i} \cdot \mathbf{P}_{i\beta}^{\mathrm{T}} \ \ \forall\, i = \overline{1,\ N-1} \tag{31}$$

and

$$\max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \mathbf{P}_{N\alpha} \cdot \mathbf{K}_{N} \cdot \mathbf{P}_{N\beta}^{\mathrm{T}} =$$

$$= \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} =$$

$$= \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)*} p_{N\beta}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \mathbf{P}_{N\alpha}^{*} \cdot \mathbf{K}_{N} \cdot \left(\mathbf{P}_{N\beta}^{*}\right)^{\mathrm{T}} =$$

$$= \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} =$$

$$= \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_{\beta}} \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_{\alpha}} \mathbf{P}_{N\alpha} \cdot \mathbf{K}_{N} \cdot \mathbf{P}_{N\beta}^{\mathrm{T}}. \tag{32}$$

So,

$$\max_{x(t) \in X} \min_{y(t) \in Y} K\left(x\left(t\right),\ y\left(t\right)\right) =$$

$$= \sum_{i=1}^{N-1} \left(\max_{\mathbf{P}_{i\alpha} \in \mathcal{P}_\alpha} \min_{\mathbf{P}_{i\beta} \in \mathcal{P}_\beta} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_\alpha} \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_\beta} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)*} p_{i\beta}^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)*} p_{N\beta}^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_{i\alpha}^{*} \cdot \mathbf{K}_i \cdot \left(\mathbf{P}_{i\beta}^{*}\right)^{\mathrm{T}} + \mathbf{P}_{N\alpha}^{*} \cdot \mathbf{K}_N \cdot \left(\mathbf{P}_{N\beta}^{*}\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N-1} \left(\min_{\mathbf{P}_{i\beta} \in \mathcal{P}_\beta} \max_{\mathbf{P}_{i\alpha} \in \mathcal{P}_\alpha} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{i\alpha}^{(m)} p_{i\beta}^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \min_{\mathbf{P}_{N\beta} \in \mathcal{P}_\beta} \max_{\mathbf{P}_{N\alpha} \in \mathcal{P}_\alpha} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_{N\alpha}^{(m)} p_{N\beta}^{(q)} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \min_{y(t) \in Y} \max_{x(t) \in X} K\left(x\left(t\right),\ y\left(t\right)\right) \tag{33}$$

and, therefore, the stack of successive solutions $\left\{\mathbf{P}_{i\alpha}^{*},\ \mathbf{P}_{i\beta}^{*}\right\}_{i=1}^{N}$ is an optimal solution in game (6). □

Obviously, owing to the solutions of the $(M+1) \times (Q+1)$ matrix games are independent, they are stacked in a way similar to how it is done by Theorem 2. If all N matrix games are solved in pure strategies, then stacking their solutions is fulfilled trivially. When there is at least an equilibrium in mixed strategies for an interval, the stacking is fulfilled as well implying that the resulting pure-mixed-strategy solution of game (6) is realized successively, interval by interval, spending the same amount of time to implement both pure strategy and mixed strategy solutions.

6. Exemplification

To exemplify how the suggested method solves zero-sum games defined on a product of staircase-function finite spaces, consider a case in which $t \in [0;\ 2\pi]$, the set of pure strategies of the first player is

$$X = \{x(t),\ t \in [0;\ 2\pi] :\ 10 \leqslant x(t) \leqslant 20\} \subset \mathbb{L}_2 [0;\ 2\pi] \tag{34}$$

and the set of pure strategies of the second player is

$$Y = \{y(t),\ t \in [0;\ 2\pi] :\ 10 \leqslant y(t) \leqslant 20\} \subset \mathbb{L}_2 [0;\ 2\pi]. \tag{35}$$

The payoff functional is

$$K(x(t),\ y(t)) = \int\limits_{[0;\ 2\pi]} \left[\sin(xt) - \sin(yt) + \cos(0.1xyt)\right] d\mu(t). \tag{36}$$

Each of the players is allowed to change its pure strategy value at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{19} = \{0.1\pi i\}_{i=1}^{19}.$$

Consequently, this game can be thought of as it is defined on rectangle (18), wherein the game is

$$\left\langle \left\{ \{9+m\}_{m=1}^{11},\ \{9+q\}_{q=1}^{11} \right\},\ \mathbf{K}_i \right\rangle \tag{37}$$

with payoff matrices $\mathbf{K}_i = [k_{imq}]_{11\times 11}$ whose elements are

$$k_{imq} = \int\limits_{[0.1\cdot(i-1)\pi;\ 0.1\pi i)} f(9+m,\ 9+q,\ t)\, d\mu(t) =$$

$$= \int\limits_{[0.1\cdot(i-1)\pi;\ 0.1\pi i)} \Big[\sin(9t+mt) - \sin(9t+qt) + \cos\big(0.1t(9+m)(9+q)\big)\Big] d\mu(t)$$

$$\text{for } i = \overline{1,\ 19} \tag{38}$$

and

$$k_{20mq} = \int\limits_{[1.9\pi;\ 2\pi]} f(9+m,\ 9+q,\ t)\, d\mu(t) =$$

$$= \int\limits_{[1.9\pi;\ 2\pi]} \Big[\sin(9t+mt) - \sin(9t+qt) + \cos\big(0.1t(9+m)(9+q)\big)\Big] d\mu(t). \tag{39}$$

The 11×11 matrix games (37) with (38) and (39) are solved in pure and mixed strategies. The stack of the 20 first player's optimal strategies in each of those 20 11×11 matrix games is shown in Figure 1, where the solid line corresponds to an optimal pure strategy and the dotted lines correspond to nonzero-probability pure strategies in an optimal mixed strategy. Similarly, the stack of the 20 second player's optimal strategies is shown in Figure 2. The process of how the first player's payoff changes as the time goes by is presented in Figure 3.

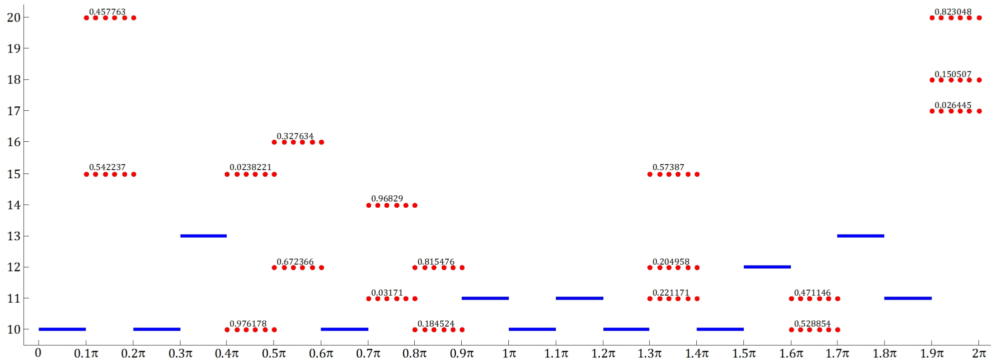

Figure 1: The stack of the 20 first player's optimal strategies

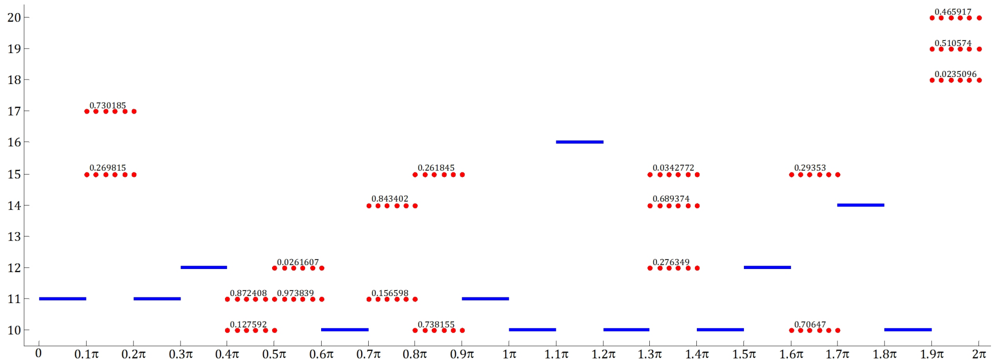

Figure 2: The stack of the 20 second player's optimal strategies

It is worth noting that the cumulative sum of the player's (either first or second) payoffs does not have to be non-decreasing (let alone increasing). This is clearly illustrated by Figure 3 as well. The first player's payoff maximum at $t = 1.9\pi$ cannot be held back from the slight decrement because in the final, 20-th (interval) game, the factual payoff of the first player is a negative amount, whereas the second player receives the positive amount. Any deviation from the optimal strategy in game (6) on product (7) by (34) — (39), say, not practicing exactly the stack in Figure 1, will not increase the cumulative sum at $t = 2\pi$ (for the first player), which is approximately 0.245389. At that, the second player cumulative loss (or payment to the first player) will not be greater than 0.245389 (if the second player sticks to the stack in Figure 2).

Another interesting peculiarity is the time of computations in this example. An 11×11 matrix game (37) with (38) and (39) is solved within 10 milliseconds on a laptop with an Intel Core i7 processor. So, the initial game is solved within 0.2 seconds

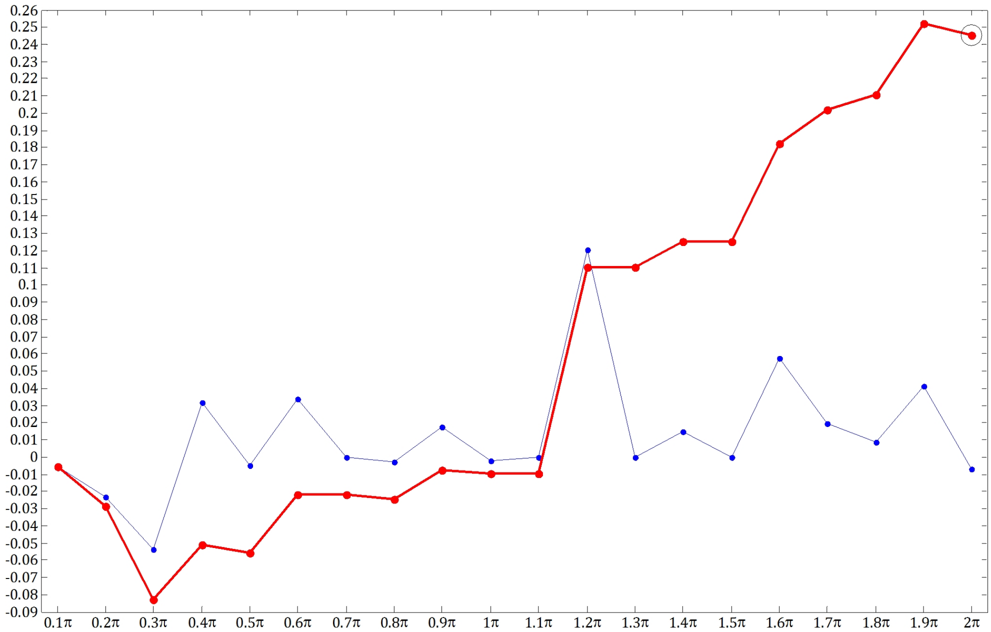

Figure 3: The first player's payoffs at the end of every interval and their cumulative sum (thicker polyline)

owing to the interval-wise solving and stacking. Surely, solving the game directly (by rendering it to the matrix game) would be quite impossible: whereas each player has 11^{20} pure strategies (in the form of staircase functions), the resulting $11^{20} \times 11^{20}$ matrix game cannot be solved in a reasonable time span.

7. Discussion

As to the computation time in general, it has an exponentially-increasing dependence on the size of the square matrix. Therefore, while a "smaller" matrix game is solved, it is presumed that the most efficient algorithm is used [8, 15]. In a pessimistic way, the applicability of the suggested method may be limited to the "smaller" matrix game size. Solving matrix games, in which a player has a few hundred pure strategies, may be time-consuming.

A peculiar issue is how to make game (6) be equivalent to matrix games. As it has been mentioned above, the players' actions can be constrained to staircase functions whose breakpoints have to be the same for both the players. This constraint is applied by the administrator of the process modeled by game (6) on (7) by (1) — (5), but what if there is no administrator (governor, regulator, manager, etc.)? This question is still open, because no theory of self-regulating antagonistic games (in which the players would develop the strategy-generating rules fit best to the game outcome) has been built yet. Nevertheless, several attempts to substantiate fundamentals of such a theory have been made [10, 13, 11, 12, 18].

One can notice that if time t is not explicitly included into the function under the integral in (5), then, if $x(t)$ and $y(t)$ are (forcedly) set to some constants on an interval, the payoff value depends only on the length of the interval. If the length does not change, every interval has the same matrix game. The triviality of the equal-length-interval solution is explained by a standstill of the players' strategies. Time variable t explicitly included into (5) means that something is going on or changes within the process as time goes by (and the players develop their actions).

After all, the presented method is a significant contribution to the antagonistic game theory. It allows solving zero-sum games with staircase-function strategies in a far simpler manner. It "deeinstellungizes" the initial game along with its solution interpretation [6]. The pure-mixed-strategy solution (like that in Figure 1 and Figure 2) can be easily implemented and practiced [9, 10].

8. Conclusion

A zero-sum game defined on a product of staircase-function finite spaces is equivalent to a matrix game. However, this game payoff matrix is built very slowly, so it is impracticable to find its solution. On the other hand, the zero-sum game is equivalent to the succession of "smaller" matrix games, each defined on an interval where the pure strategy value is constant. Thus, owing to Theorem 3, the solution of the initial game can be obtained by stacking the solutions of the "smaller" matrix games. The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is continuous). Moreover, any combination of the solutions of the "smaller" matrix games is a solution of the initial zero-sum game.

Solving games on a product of staircase-function finite spaces can be studied also for the case of non-antagonistic interests of two players. Nonetheless, an approach to solving the corresponding bimatrix games is not straightforwardly deduced from the suggested method. The matter is the optimality in the matrix game does not have an analogy for the bimatrix game [14, 15]. This specificity will make a generalized study of two-person games dissimilar to the presented study.

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On Multiplicative (Generalized)-Derivations and Central Valued Conditions in Prime Rings

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ABSTRACT: Let R be a prime ring with multiplicative (generalized)-derivations (F, f) and (G, g) on R. This paper gives a number of central valued algebraic identities involving F and G that are equivalent to the commutativity of R under some suitable assumptions. Moreover, in order to optimize our results, we show that the assumptions taken cannot be relaxed.

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Keywords and Phrases: Prime ring; Multiplicative generalized derivation; One-sided ideal; Commutativity.

1. Introduction

All through this paper, R will always denote an associative prime ring, unless otherwise mentioned and $Z(R)$ denotes the center of R. For any $x, y \in R$, the commutator $xy - yx$ is denoted by the symbol $[x, y]$. It is well-known that a ring R is said to be prime if for any $a, b \in R$; $aRb = (0)$ implies either $a = 0$ or $b = 0$ and it is called semiprime if $aRa = (0)$ implies $a = 0$. By a multiplicative derivation, we mean a mapping $\delta : R \to R$ (not necessarily additive) satisfying the relation $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. If δ is necessarily additive, then it is called a derivation. Therefore every derivation is a multiplicative derivation but the converse is not generally true; for example, let $R = C[0, 1]$ be the ring of all continuous (real or complex valued) functions and define a map $\delta : R \to R$ such that

$$\delta(f)(x) = \begin{cases} f(x)\log|f(x)|, & \text{when } f(x) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation δ of R such that $F(xy) = F(x)y + x\delta(y)$ for all $x, y \in R$. Then it is natural to think of the unified notion of generalized derivation and multiplicative derivation. Recently, Dhara and Ali [7] introduced a mapping called multiplicative (generalized)-derivation viz., a mapping $F : R \to R$ is called a multiplicative (generalized)-derivation if there exists a function $f : R \to R$ such that $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$, where F and f are not necessarily additive. However, these mappings appeared first time in a paper by Gusić [9]. Later on it is figured out that the associated function of a multiplicative (generalized)-derivation is a multiplicative derivation (see [6]).

During the last six decades there have been many results showing that the global structure of a ring is often tightly connected with the behaviour of additive and multiplicative mappings defined on it (see [2], [9], [10], [11], [12], [13]). In 1957, Posner [11] initiated the study of identities involving derivations that ensure commutativity. More precisely, the classical Posner's second theorem states that a prime ring must be commutative if it admits a non-zero derivation d satisfying $d(x)x - xd(x) \in Z(R)$ for all $x \in R$. In this direction, Ashraf and Rehman [3] examined the commutativity of a prime ring R admitting a non-zero derivation d that satisfies any one of the conditions: $d(xy) \pm xy \in Z(R)$, $d(xy) \pm yx \in Z(R)$ and $d(x)d(y) \pm xy \in Z(R)$ for all $x, y \in I$, a non-zero ideal of R. These results have been proved for generalized derivations in [4]. In [1], Albaş proved the following theorem: *Let R be a prime ring with $char(R) \neq 2$. If R admits a generalized derivation F with associated derivation d satisfying $F([x, y]) = [F(x), F(y)]$ or $F([x, y]) = [F(y), F(x)]$ for all $x, y \in R$, then either R is commutative or $d = 0$ or $d = 1_{id}$ or $d = -1_{id}$, where 1_{id} is the identity map of R.* Very recently, Huang [10] explored the commutativity of prime rings with specific additive mapping F that satisfy the following identities: (i) $F(xy) \pm xy \in Z(R)$, (ii) $F([x, y]) \pm [F(x), y] \in Z(R)$, (iii) $F([x, y]) \pm [F(x), F(y)] \in Z(R)$, (iv) $[F(x), y] \pm [x, F(y)] \in Z(R)$. Moreover, it would also be interesting and general to investigate the commutativity of prime and semiprime rings with various types of multiplicative derivations. Gusić [9] proved that if F is a multiplicative (generalized)-derivation of a prime ring R associated with a non-zero mapping d and I is a non-zero ideal of R such that $F(xy) = F(y)F(x)$ for all $x, y \in I$, then R is commutative. In 2013, Dhara and Ali [7] collected information about the commutative structure of prime and semiprime rings admitting multiplicative (generalized)-derivations that satisfy any one of the conditions: (i) $F(xy) \pm xy \in Z(R)$, (ii) $F(xy) \pm yx \in Z(R)$, (iii) $F(x)F(y) \pm xy \in Z(R)$, (iv) $F(x)F(y) \pm yx \in Z(R)$. In the same line of investigation, Ali et al. [2] obtained many structural results of prime and semiprime rings. Recently, Dhara and Pradhan [8] studied the following left annihilator conditions: (i) $a(F(xy) \pm xy) = 0$, (ii) $a(F(xy) \pm yx) = 0$, (iii) $a(d(x)F(y) \pm xy) = 0$, (iv) $a(d(x)F(y) \pm yx) = 0$, (v) $a(F(x)F(y) \pm xy) = 0$, (vi) $a(F(x)F(y) \pm yx) = 0$ for all $x, y \in I$, where F is a generalized (multiplicative)-derivation, d is the associated multiplicative derivation and I is a non-zero ideal of a prime ring R.

It is known that every noncentral Jordan ideal and every noncentral square-closed Lie ideal of a 2-torsion free semiprime ring R contains a non-zero ideal of R (see [13], [14] resp.). Therefore, in this view it is the optimal case to study certain algebraic identities over one-sided ideals of rings. In the present paper, motivated by Huang [10] and Dhara and Pradhan [8], we study a number of central valued conditions with multiplicative (generalized)-derivations on one-sided ideals of prime rings and obtain the commutativity of R.

2. Preliminary Results

The following basic commutator identities are useful in the sequel.

$$[xy, z] = x\,[y, z] + [x, z]\,y, [x, yz] = y\,[x, z] + [x, y]\,z.$$

We begin our discussion with the following basic lemmas that will be frequently used in our results.

Lemma 2.1 ([10], Lemma 2). *Let R be a prime ring. Then for some $0 \neq a \in Z(R)$, if $ab \in Z(R)$, then $b \in Z(R)$. In particular, if $ab = 0$, then $b = 0$.*

Lemma 2.2 ([5]). *Every prime ring R having a non-zero one sided commutative ideal is commutative.*

Lemma 2.3. *Let R be a prime ring and I be a non-zero left (resp. right) ideal of R. If for any $a, b \in R$, $aIb = (0)$, then either $a = 0$ or $Ib = (0)$ (resp. $aI = (0)$ or $b = 0$).*

Proof. Since R is a prime ring, this fact can be easily obtained. □

Lemma 2.4. *Let R be a ring and δ be a multiplicative derivation of R. Then $\delta(Z(R)) \subseteq Z(R)$.*

Proof. It is trivial to observe that $\delta(0) = 0$. Now let $c \in Z(R)$ be any element.

$$\delta(xc) = \delta(x)c + x\delta(c) \text{ and } \delta(cx) = \delta(c)x + c\delta(x).$$

Combining these both expressions, we get $[\delta(c), x] = 0$ for all $x \in R$. Hence $\delta(c) \in Z(R)$ for all $c \in Z(R)$. □

Lemma 2.5. *Let R be a prime ring and ϱ be a non-zero right ideal of R. If there exists $a \in R$ such that $a\varrho \neq (0)$ and $a[x, y] \in Z(R)$ for all $x, y \in \varrho$, then R is commutative.*

Proof. Let us assume that $a[x, y] \in Z(R)$ for all $x, y \in \varrho$. It means $[a\,[x, y]\,, r] = 0$ for all $x, y \in \varrho$ and $r \in R$. It implies that

$$a\,[[x, y]\,, r] + [a, r]\,[x, y] = 0, \ \forall\ x, y \in \varrho,\ r \in R.$$

Take yx instead of y in above relation to obtain $a\,[x, y]\,[x, r] = 0$ for all $x, y \in \varrho$ and $r \in R$. It implies that $a\,[x, y]\,R\,[x, r] = (0)$ for all $x, y \in \varrho$ and $r \in R$. Thus for each $x \in \varrho$, either $a\,[x, \varrho] = (0)$ or $x \in Z(R)$. Therefore in each case we have

$$a\,[x, y] = 0, \ \forall\ x, y \in \varrho.$$

Take xs for x in last expression, where $s \in R$, we get $ax\,[s, y] = 0$. Since $a\varrho \neq (0)$, by Lemma 2.3, it is straightforward to find that $\varrho \subseteq Z(R)$. Hence R is commutative, by Lemma 2.2. □

Lemma 2.6. *Let R be a prime ring and ς be a non-zero left ideal of R. If $\sigma : R \to R$ is a ring automorphism such that $\sigma([x,y]) \in Z(R)$ for all $x, y \in \varsigma$, then R is commutative.*

Proof. Let us assume that $\sigma([x,y]) \in Z(R)$ for all $x, y \in \varsigma$. It implies that

$$[\sigma([x,y]), r] = 0, \ \forall \ x, y \in \varsigma, \ r \in R.$$

Take xy for y in this equation, we find

$$[\sigma(x), r]\,[\sigma(x), \sigma(y)] = 0, \ \forall \ x, y \in \varsigma, \ r \in R.$$

Replace r by sr such that $s \in R$ in the above expression, we obtain

$$[\sigma(x), s]\,R\,[\sigma(x), \sigma(y)] = (0), \ \forall \ x, y \in \varsigma, \ s \in R.$$

It implies that for each $x \in \varsigma$, either $[\sigma(x), \sigma(\varsigma)] = (0)$ or $\sigma(x) \in Z(R)$. Hence $[\sigma(x), \sigma(y)] = 0$ for all $x, y \in \varsigma$. In view of Lemma 2.2, R is commutative. □

3. Main Results

Theorem 3.1. *Let R be a prime ring and ϱ be a non-zero right ideal of R. Suppose that R admits multiplicative (generalized)-derivations (F, f) and (G, g) such that one of f and g is non-vanishing on $Z(R)$. If there exists $a \in R$ such that $a\varrho \neq (0)$, then the following assertions are equivalent:*

(i) $a([F(x), y] \pm [x, G(y)]) \in Z(R)$ *for all* $x, y \in \varrho$.

(ii) $a(F([x, y]) \pm [G(x), y]) \in Z(R)$ *for all* $x, y \in \varrho$.

(iii) $a(F([x, y]) \pm [x, G(y)]) \in Z(R)$ *for all* $x, y \in \varrho$.

(iv) R *is commutative.*

Proof. $(i) \Rightarrow (iv)$: Let us first assume that

$$a([F(x), y] + [x, G(y)]) \in Z(R), \ \forall \ x, y \in \varrho. \tag{3.1}$$

In case $(0) \neq f(Z(R))$, we choose $c \in Z(R)$ such that $0 \neq f(c)$ and replace x by xc in (3.1) to get

$$a([F(x), y] + [x, G(y)])c + a\,[xf(c), y] \in Z(R), \ \forall \ x, y \in \varrho.$$

Eq. (3.1) gives $a\,[xf(c), y] \in Z(R)$ for all $x, y \in \varrho$. Since f is a multiplicative derivation (see [6, Lemma 2]), in view of Lemma 2.4, it follows that $a\,[x, y]\,f(c) \in Z(R)$ for all $x, y \in \varrho$. Applying Lemma 2.1, it implies that $a\,[x, y] \in Z(R)$ for all $x, y \in \varrho$. By Lemma 2.5, R is commutative.

Let us now assume that $0 \neq g(Z(R))$. Then we choose $c' \in Z(R)$ such that $0 \neq g(c')$. Substitute yc' in place of y in (3.1) to find

$$a([F(x), y] + [x, G(y)])c' + a\,[x, yg(c')] \in Z(R),\ \forall\ x, y \in \varrho.$$

It implies $a\,[x, y]\,g(c') \in Z(R)$ for all $x, y \in \varrho$. With similar computations as in the case $(0) \neq f(Z(R))$, we get the conclusion.

Notice that if (G, g) is a multiplicative (generalized)-derivation, then so is $(-G, -g)$. Thus by taking $(-G, -g)$ in place of (G, g) in the above proof, we get the same outcome with the identity $a([F(x), y] - [x, G(y)]) \in Z(R)$ for all $x, y \in \varrho$.

$(ii) \Rightarrow (iv)$: Let us assume that

$$a(F([x, y]) + [G(x), y] \in Z(R),\ \forall\ x, y \in \varrho. \tag{3.2}$$

Choose $c \in Z(R)$ such that $0 \neq f(c)$, take yc for y in (3.2) we get

$$a(F([x, y]) + [G(x), y])c + a\,[x, y]\,f(c) \in Z(R),\ \forall\ x, y \in \varrho.$$

It implies that $a\,[x, y]\,f(c) \in Z(R)$ for all $x, y \in \varrho$. Since f is a multiplicative derivation, in view of Lemma 2.4 and Lemma 2.1, we find that $a\,[x, y] \in Z(R)$ for all $x, y \in \varrho$. By Lemma 2.5, we get the conclusion.

We now assume that $0 \neq g(Z(R))$. Then we choose $c' \in Z(R)$ such that $0 \neq g(c')$. Substitute yc' in place of y in (3.2), we obtain

$$a(F([x, y]) + [G(x), y])c' + a\,[x, y]\,f(c') \in Z(R),\ \forall\ x, y \in \varrho.$$

Using the hypothesis, we get

$$a\,[x, y]\,f(c') \in Z(R),\ \forall\ x, y \in \varrho. \tag{3.3}$$

Substitute xc' in place of x in (3.2), we find that

$$a(F([x, y]) + [G(x), y])c' + a\,[x, y]\,f(c') + a\,[xg(c'), y] \in Z(R),\ \forall\ x, y \in \varrho.$$

Using the hypothesis, we get

$$a\,[x, y]\,f(c') + a\,[xg(c'), y] \in Z(R),\ \forall\ x, y \in \varrho. \tag{3.4}$$

Eq. (3.4) and (3.3) gives $a\,[xg(c'), y] \in Z(R)$ for all $x, y \in \varrho$. Since g is a multiplicative derivation, we find $a[x, y]g(c') \in Z(R)$ for all $x, y \in \varrho$. Thus, it follows that $a\,[x, y] \in Z(R)$ for all $x, y \in \varrho$. By Lemma 2.5, we get the conclusion. In the same way, we get the desired result with $a(F([x, y]) - [G(x), y]) \in Z(R)$ for all $x, y \in \varrho$.

$(iii) \Rightarrow (iv)$: Let us consider the situation

$$a(F([x, y]) + [x, G(y)]) \in Z(R),\ \forall\ x, y \in \varrho. \tag{3.5}$$

In case $(0) \neq f(Z(R))$, we choose $c \in Z(R)$ such that $0 \neq f(c)$ and replace x by xc in (3.5) to get

$$a(F([x, y]) + [x, G(y)])c + a\,[x, y]\,f(c) \in Z(R),\ \forall\ x, y \in \varrho.$$

Eq. (3.5) gives $[x,y]\,f(c) \in Z(R)$ for all $x,y \in \varrho$ and $c \in Z(R)$. Since f is a multiplicative derivation, we find $a\,[x,y] \in Z(R)$ for all $x,y \in \varrho$. Thus R is commutative, by Lemma 2.5.

Let us now assume that $(0) \neq g(Z(R))$. Then we choose $c' \in Z(R)$ such that $0 \neq g(c')$. Substitute yc' in place of y in (3.5), we find

$$a(F([x,y]) + [x,G(y)])c' + a\,[x,y]\,f(c') + a\,[x,yg(c')] \in Z(R),\ \forall\ x,y \in \varrho.$$

Using the hypothesis, we get

$$a\,[x,y]\,f(c') + a\,[x,yg(c')] \in Z(R),\ \forall\ x,y \in \varrho. \tag{3.6}$$

Substitute xc' in place of x in (3.5), we find that

$$a(F([x,y]) + [x,G(y)])c' + a\,[x,y]\,f(c') \in Z(R),\ \forall\ x,y \in \varrho.$$

Using the hypothesis, we get

$$a\,[x,y]\,f(c') \in Z(R),\ \forall\ x,y \in \varrho. \tag{3.7}$$

Eq. (3.6) and (3.7) gives $a\,[x,y]\,g(c') \in Z(R)$ for all $x,y \in \varrho$. Since g is a multiplicative derivation, $a\,[x,y] \in Z(R)$ for all $x,y \in \varrho$. In view of Lemma 2.5, R is commutative. In the same way, we get the desired conclusion from $a(F([x,y]) - [x,G(y)] \in Z(R)$ for all $x,y \in \varrho$. □

Corollary 3.2. *Let R be a prime ring and I be a non-zero ideal of R. Suppose that R admits multiplicative (generalized)-derivations (F,f) and (G,g) such that one of f and g is non-vanishing on $Z(R)$. For some $0 \neq a \in R$, the following assertions are equivalent:*

(i) $a([F(x),y] \pm [x,G(y)]) \in Z(R)$ *for all* $x,y \in I$.

(ii) $a(F([x,y]) \pm [G(x),y]) \in Z(R)$ *for all* $x,y \in I$.

(iii) $a(F([x,y]) \pm [x,G(y)]) \in Z(R)$ *for all* $x,y \in I$.

(iv) *R is commutative.*

Proof. The proof is straightforward. □

Theorem 3.3. *Let R be a prime ring and ς be a non-zero left ideal of R. Suppose that R admits multiplicative (generalized)-derivations $(F,f),(G,g)$ and a ring automorphism σ such that one of f and g is non-vanishing on $Z(R)$. Then the following assertions are equivalent:*

(i) $\sigma\,([F(x),y] \pm [x,G(y)]) \in Z(R)$ *for all* $x,y \in \varsigma$.

(ii) $\sigma\,(F([x,y]) \pm [G(x),y]) \in Z(R)$ *for all* $x,y \in \varsigma$.

(iii) $\sigma\,(F([x,y]) \pm [x,G(y)]) \in Z(R)$ *for all* $x,y \in \varsigma$.

(iv) R is commutative.

Proof. $(i) \Rightarrow (iv)$: Let us consider the situation

$$\sigma\left([F(x), y]+[x, G(y)]\right) \in Z(R),\ \forall\ x, y \in \varsigma. \tag{3.8}$$

In case $(0) \neq f(Z(R))$, we choose $c \in Z(R)$ such that $0 \neq f(c)$ and replace x by cx in (3.8) to get

$$\sigma\left([F(x), y]+[x, G(y)]\right)\sigma(c)+\sigma\left([x f(c), y]\right) \in Z(R),\ \forall\ x, y \in \varsigma. \tag{3.9}$$

Eq. (3.9) gives that $\sigma\left([x f(c), y]\right) \in Z(R)$ for all $x, y \in \varsigma$. In view of Lemma 2.4, we have $\sigma([x, y])\sigma(f(c)) \in Z(R)$ for all $x, y \in \varsigma$. Since $0 \neq f(c)$ and σ is injective, so $\sigma(f(c)) \neq 0$. It follows from Lemma 2.1 that

$$\sigma([x, y]) \in Z(R),\ \forall\ x, y \in \varsigma. \tag{3.10}$$

By Lemma 2.6, R is commutative.

We now assume that $(0) \neq g(Z(R))$. Let us choose $c' \in Z(R)$ such that $0 \neq g(c')$. Replace y with $c'y$ in (3.8), we get

$$\sigma\left([F(x), y]+[x, G(y)]\right)\sigma(c')+\sigma([x, y g(c')]) \in Z(R),\ \forall\ x, y \in \varsigma.$$

Using the hypothesis, we obtain $\sigma([x, y g(c')]) \in Z(R)$ for all $x, y \in \varsigma$. In view of Lemma 2.4, we have

$$\sigma([x, y])\sigma(g(c')) \in Z(R),\ \forall\ x, y \in \varsigma.$$

Since $0 \neq \sigma(g(c'))$, we get $\sigma([x, y]) \in Z(R)$ for all $x, y \in \varsigma$. In view of Lemma 2.6, we get our conclusion. In the same way, we get commutativity of R from the condition $\sigma([F(x), y]-[x, G(y)]) \in Z(R)$ for all $x, y \in \varsigma$.

$(ii) \Rightarrow (iv)$: We now assume that

$$\sigma\left(F([x, y])+[G(x), y]\right) \in Z(R),\ \forall\ x, y \in \varsigma. \tag{3.11}$$

Choose $c \in Z(R)$ such that $0 \neq f(c)$, take yc for y in (3.11) in order to obtain

$$\sigma\left(F([x, y])+[G(x), y]\right)\sigma(c)+\sigma\left([x, y]\right)\sigma\left(f\left(c\right)\right) \in Z(R),\ \forall\ x, y \in \varsigma.$$

Eq. (3.11) gives $\sigma\left([x, y]\right)\sigma\left(f\left(c\right)\right) \in Z(R)$ for all $x, y \in \varsigma$. By same reasoning as above, we have $\sigma\left([x, y]\right) \in Z(R)$ for all $x, y \in \varsigma$. By Lemma 2.6, R is commutative.

Assume that $(0) \neq g(Z(R))$. By our assumption there exists $c' \in Z(R)$ such that $0 \neq g(c')$. Replace y by yc' in (3.11), we get $\sigma\left(F([x, y])+[G(x), y]\right)\sigma\left(c'\right)+\sigma\left([x, y]\right)\sigma\left(f\left(c'\right)\right) \in Z(R)$ for all $x, y \in \varsigma$. By the given hypothesis, we have $\sigma\left([x, y]\right)\sigma\left(f\left(c'\right)\right) \in Z(R)$ for all $x, y \in \varsigma$. Take xc' instead of x in (3.11) and using it, we may infer that $\sigma\left([x, y]\right)\sigma\left(f\left(c'\right)\right)+\sigma\left([x g\left(c'\right), y]\right) \in Z(R)$ for all $x, y \in \varsigma$. Further it reduces to $\sigma\left([x g\left(c'\right), y]\right) \in Z(R)$ for all $x, y \in \varsigma$. Applying Lemma 2.4, we get $\sigma\left([x, y]\right)\sigma\left(g\left(c'\right)\right) \in Z(R)$ for all $x, y \in \varsigma$. In view of Lemma 2.1, we obtain

$\sigma([x,y]) \in Z(R)$ for all $x,y \in \varsigma$. Hence R is commutative, by Lemma 2.6. Similarly, we can get the results when $\sigma(F([x,y]-[G(x),y])) \in Z(R)$ for all $x,y \in \varsigma$.

$(iii) \Rightarrow (iv)$: Let us consider

$$\sigma\left(F([x,y]) + [x,G(y)]\right) \in Z(R),\ \forall\ x,y \in \varsigma. \tag{3.12}$$

In case $(0) \neq g(Z(R))$, we choose $c \in Z(R)$ such that $0 \neq g(c)$ and replace y by yc in (3.12) we have

$$\sigma\left(F([x,y]) + [x,G(y)]\right)\sigma(c) + \sigma([x,y])\,\sigma(f(c)) + \sigma([x,yg(c)]) \in Z(R),\ \forall\ x,y \in \varsigma. \tag{3.13}$$

Substitute xc in place of x in (3.12), we find that

$$\sigma\left(F([x,y]) + [x,G(y)]\right)\sigma(c) + \sigma([x,y])\,\sigma(f(c)) \in Z(R),\ \forall\ x,y \in \varsigma. \tag{3.14}$$

Eq. (3.13) and (3.14) gives $\sigma([x,y])\,\sigma(g(c)) \in Z(R)$ for all $x,y \in \varrho$. It follows from above that $\sigma([x,y]) \in Z(R)$ for all $x,y \in \varrho$. With the aid of Lemma 2.6, we are done.

We next assume that $(0) \neq f(Z(R))$. One may notice that the similar implications as in the above case ensure the conclusion, therefore we omit the details. In the same way, we get the conclusion from $\sigma(F([x,y]) - [x,G(y)]) \in Z(R)$ for all $x,y \in \varsigma$. □

With the similar arguments as in the proof of Theorem 3.1 with necessary modifications, we obtain the following result and for the sake of brevity, we omit its proof.

Theorem 3.4. *Let R be a prime ring and ϱ be a non-zero right ideal of R. Suppose that R admits multiplicative (generalized)-derivations (F,f) and (G,g) such that f and g are both non-vanishing on $Z(R)$. If there exists some $a \in R$ such that $a\varrho \neq (0)$, then the following assertions are equivalent:*

(i) $a\left(F([x,y]) \pm [G(x),F(y)]\right) \in Z(R)$ *for all* $x,y \in \varrho$.

(ii) $a\left(F([x,y]) \pm [F(x),G(y)]\right) \in Z(R)$ *for all* $x,y \in \varrho$.

(iii) R is commutative.

Corollary 3.5. *Let R be a prime ring and I be a non-zero ideal of R. Suppose that R admits multiplicative (generalized)-derivations (F,f) and (G,g) such that f and g are both non-vanishing on $Z(R)$. Then for some $0 \neq a \in R$, the following assertions are equivalent:*

(i) $a\left(F([x,y]) \pm [G(x),F(y)]\right) \in Z(R)$ *for all* $x,y \in I$.

(ii) $a\left(F([x,y]) \pm [F(x),G(y)]\right) \in Z(R)$ *for all* $x,y \in I$.

(iii) R is commutative

Theorem 3.6. *Let R be a prime ring and ς be a non-zero left ideal of R. Suppose that R admits a ring automorphism σ and multiplicative (generalized)-derivations (F,f) and (G,g) such that f and g are both non-vanishing on $Z(R)$. Then the following assertions are equivalent:*

(i) $\sigma\left(F\left([x, y]\right) \pm [G(x), F(y)]\right) \in Z(R)$ *for all* $x, y \in \varsigma$.

(ii) $\sigma\left(F\left([x, y]\right) \pm [F(x), G(y)]\right) \in Z(R)$ *for all* $x, y \in \varsigma$.

(iii) R *is commutative.*

Proof. $(i) \Rightarrow (iii)$: Let us first consider the situation $\sigma\left(F\left([x, y]\right) + [G(x), F(y)]\right) \in Z(R)$ for all $x, y \in \varsigma$. By applying the same technique that we used in the proof of Theorem 3.4, we get $\sigma([x, y]) \in Z(R)$ for all $x, y \in \varsigma$. By Lemma 2.6, we are done. Similarly, we conclude in case $\sigma\left(F\left([x, y]\right) - [G(x), F(y)]\right) \in Z(R)$ for all $x, y \in \varsigma$.

$(ii) \Rightarrow (iii)$: Proof is omitted. □

Theorem 3.7. *Let R be a prime ring and ς be a non-zero left ideal of R. Suppose that R admits multiplicative (generalized)-derivations (F, f) and (G, g) such that both f and g are non-vanishing on $Z(R)$. Then for some $0 \neq a \in R$, the following assertions are equivalent:*

(i) $a\left(F(x)G(y) \pm xy\right) \in Z(R)$ *for all* $x, y \in \varsigma$.

(ii) $a\left(F(x)G(y) \pm yx\right) \in Z(R)$ *for all* $x, y \in \varsigma$.

(iii) R *is commutative.*

Proof. $(i) \Rightarrow (iii)$: We first consider the case

$$a\left(F(x)G(y) + xy\right) \in Z(R), \ \forall\ x, y \in \varsigma. \tag{3.15}$$

In view of our assumption, let us choose $c \in Z(R)$ such that $0 \neq g(c)$. Take yc for y in (3.15) to get

$$a\left(F(x)G(y) + xy\right)c + aF(x)yg(c) \in Z(R), \ \forall\ x, y \in \varsigma.$$

By the given hypothesis, it implies that $aF(x)yg(c) \in Z(R)$ for all $x, y \in \varsigma$. Since $0 \neq g(c)$, by Lemma 2.1, we have $aF(x)y \in Z(R)$ for all $x, y \in \varsigma$. That is

$$[aF(x)y, r] = 0, \ \forall\ x, y \in \varsigma, r \in R. \tag{3.16}$$

Take xt in place of x in (3.16), we obtain

$$[aF(x)ty, r] + [axf(t)y, r] = 0, \ \forall\ x, y, t \in \varsigma, r \in R.$$

Eq. (3.16) reduces it to

$$[axf(t)y, r] = 0, \ \forall\ x, y, t \in \varsigma, r \in R.$$

Replace x by ax in the above expression and using it to get

$$[a, r]\, axf(t)y = 0, \ \forall\ x, y, t \in \varsigma, r \in R.$$

It implies that $[a, r]\, Raxf(t)y = (0)$ for all $x, y, t \in \varsigma$ and $r \in R$. Thus we either have $a \in Z(R)$ or $axf(t)y = 0$ for all $x, y, t \in \varsigma$. Suppose that $axf(t)y = 0$ for all $x, y, t \in \varsigma$. Since $0 \neq a$, it implies that $xf(t)y = 0$ and hence $xf(t) = 0$ for all $x, t \in \varsigma$. Choose $c_1 \in Z(R)$ such that $f(c_1) \neq 0$ and substitute tc_1 instead of t, we have $c_1xf(t) + xf(c_1)t = 0$ for all $x, t \in \varsigma$. Then it is not difficult to see that $xf(c_1) = 0$ for all $x \in \varsigma$. By Lemma 2.4, we have $f(c_1)x = 0$. Take rx for x, where $r \in R$, we get that $f(c_1)Rx = (0)$ for all $x \in \varsigma$. It implies that either $f(c_1) = 0$ or $x = 0$, but none of them is true, and hence a contradiction follows.

Thus we have $a \in Z(R)$. By Lemma 2.1, from (3.15) we find that $F(x)G(y)+xy \in Z(R)$ for all $x, y \in \varsigma$. By repeating the similar arguments as above, we arrive at the situation $[xf(t)y, r] = 0$ for all $x, y, t \in \varsigma$ and $r \in R$. Replace x with qx, where $q \in R$, we obtain that $[q, r]\, xf(t)y = 0$ for all $x, y, t \in \varsigma$ and $r, q \in R$. It implies that $[q, r]\, Rxf(t)y = (0)$, and hence either $[q, r] = 0$ for all $r, q \in R$ or $xf(t)y = 0$ for all $x, y, t \in \varsigma$. Clearly the latter case is not possible, thus we have $[R, R] = (0)$. Similarly, we can conclude that R is commutative in case $a\,(F(x)G(y) - xy) \in Z(R)$ for all $x, y \in \varsigma$.

$(ii) \Rightarrow (iii)$: With the similar implications, we can easily prove this part, therefore we omit its proof. $\square$

We conclude this discussion with the following example, which shows that the hypotheses of our results are not superfluous.

Example 3.8. Let S be a ring. Consider

$$R = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a, b, c, d, e \in S \right\}.$$

Define maps $F, f, G, g : R \to R$ by

$$F\left(\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & b^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & bd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G\left(\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & bd^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$g\left(\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & abc \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be easily verified that (F, f) and (G, g) are multiplicative (generalized)-derivations of R. Note that

$$Z(R) = \left\{ \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid c \in S \right\}$$

and the set

$$\varrho = \left\{ \begin{bmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid b, c \in S \right\},$$

is a right ideal of R. One can check that none of our hypotheses are true, i.e., R is not a prime ring, f and g are both vanishing on $Z(R)$ and there exists $x \in \varrho$ such that $ax = 0$ for all $a \in R$. We note that the following identities are satisfied:

(i) $a\left([F(x), y] \pm [x, G(y)]\right) \in Z(R)$.

(ii) $a\left([F(x), y] \pm [G(x), y]\right) \in Z(R)$.

(iii) $a\left(F([x, y]) \pm [x, G(y)]\right) \in Z(R)$.

(iv) $a\left(F([x, y]) \pm [G(x), F(y)]\right) \in Z(R)$.

(v) $a\left(F([x, y]) \pm [F(x), G(y)]\right) \in Z(R)$.

(vi) $a\left(F(x)G(y) \pm xy\right) \in Z(R)$.

(vii) $a\left(F(x)G(y) \pm yx\right) \in Z(R)$.

for all $x, y \in \varrho$. However, R is not commutative.

4. Open Problems

The following are some natural questions, that we are unable to answer at that moment:

1. How to remove the condition that the associated derivations are non-vanishing on $Z(R)$?

2. Is there any example which shows that this assumption can not be relaxed to get the given results?

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Difference Sequence Spaces Defined by Musielak-Orlicz Function

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Abstract: The purpose of this paper is to introduce sequence spaces $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]_{\theta}$. We also examine some topological properties and prove some inclusion relations between these spaces.

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1. Introduction and Preliminaries

The notion of the difference sequence space was introduced by Kızmaz [1]. It was further generalized by Et and Çolak [2] as follows: $Z(\Delta^{\mu}) = \{x = (x_k) \in \omega : (\Delta^{\mu} x_k) \in z\}$ for $z = \ell_{\infty},\ c$ and $c_{\circ}$, where μ is a non-negative integer and

$$\Delta^{\mu} x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1},\ \Delta^{\circ} x_k = x_k \ \text{ for all } k \in \mathbb{N}$$

or equivalent to the following binomial representation:

$$\Delta^{\mu} x_k = \sum_{v=0}^{\mu} (-1)^v \tbinom{\mu}{v} x_{k+v}.$$

These sequence spaces were generalized by Et and Basaşir [3] taking $z = \ell_{\infty}(p)$, $c(p)$ and $c_{\circ}(p)$. Dutta [4] introduced the following difference sequence spaces using a new difference operator:

$$Z(\Delta_{(\eta)}) = \big\{x = (x_k) \in \omega : \Delta_{(\eta)} x \in z\big\} \ \text{ for } z = \ell_{\infty},\ c \text{ and } c_{\circ},$$

where $\Delta_{(\eta)}x = (\Delta_{(\eta)}x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.
In [5], Dutta introduced the sequence spaces $\bar{c}(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $\bar{c_\circ}(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $\ell_\infty(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $m(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$ and $m_\circ(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, where $\eta,\ \mu \in \mathbb{N}$ and $\Delta^{\mu}_{(\eta)}x_k = (\Delta^{\mu}_{(\eta)}x_k) = (\Delta^{\mu-1}_{(\eta)}x_k - \Delta^{\mu-1}_{(\eta)}x_{k-\eta})$ and $\Delta^{\circ}_{(\eta)}x_k = x_k$ for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^{\mu}_{(\eta)}x = \sum_{v=0}^{\mu}(-1)^v \tbinom{\mu}{v} x_{k-\eta v}.$$

The difference sequence space have been studied by authors ([6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) and references therein. Başar and Altay [16] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator, by

$$b_{mk} = \begin{cases} r, & k = m \\ s, & k = m-1 \\ 0, & (k > m) \text{ or } (0 \leq k < m-1). \end{cases}$$

Başarir and Kayikçi [17] defined the matrix $B^\mu(b^\mu_{mk})$ which reduced the difference matrix $\Delta^{\mu}_{(1)}$ incase $r = 1,\ s = -1$. The generalized B^μ-difference operator is equivalent to the following binomial representation:

$$B^\mu x = B^\mu(x_k) = \sum_{v=0}^{\mu} \tbinom{\mu}{v} r^{\mu-v} s^v x_{k-v}.$$

Let $\wedge = (\wedge_k)$ be a sequence of non-zero scalars. Then, for a sequence space E, the multiplier sequence space $E_\wedge$, associated with the multiplier sequence $\wedge$, is defined as

$$E_\wedge = \{x = (x_k) \in \omega : (\wedge_k x_k) \in E\}.$$

An Orlicz function M is a function, which is continuous non-decreasing and convex with $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Linderstrauss and Tzafriri [18] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [18] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq KLM(x)$ for all values of

$x \geq 0$ and for $L > 1$. An Orlicz function M can always be represented in the following interval form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([19], [20]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, ...$$

is called the complimentary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \ x = (x_k) \in t_M.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\{k > 0 : I_M(x/k) \leq 1\}$$

or equipped with the Orlicz norm

$$\|x\|^{\circ} = \inf\left\{\frac{1}{k}\big(1 + I_M(kx)\big) : k > 0\right\}.$$

The concept of 2-normed spaces was initially developed by Gähler [21] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan ([23], [24]) and Gunawan and Mashadi [25]. Let $n \in \mathbb{N}$ and X be linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$.

A real valued function $\|., ..., .\|$ on X^n satisfying the following four conditions:
(1) $\|x_1, x_2, .., x_n\| = 0$ if and only if $x_1, x_2, x_3, ...x_n$ are linearly dependent in X;
(2) $\|x_1, x_2, .., x_n\|$ is invariant under permutation ;
(3) $\|\alpha x_1, x_2, .., x_n\| = |\alpha|\|x_1, x_2, .., x_n\|$ for any $\alpha \in \mathbb{K}$ and
(4) $\|x + x', x_2, .., x_n\| \leq \|x, x_2, .., x_n\| + \|x', x_2, .., x_n\|$

is called an n-norm on X and the pair $(X, \|., .., .\|)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean

n-norm $\|x_1, x_2, .., x_n\|_E$ as the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, .., x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, .., x_n\|_E = |det(x_{ij})|;$$

where $x_i = (x_{i1}, x_{i2}, x_{i3}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, 3, ..., n$ and $\|.\|_E$ denotes the Euclidean norm. Let $(X, \|., .., .\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be linearly independent set in X. Then the following function $\|., ..., .\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, .., x_n\|_\infty = \max\Big\{\|x_1, x_2, .., x_{n-1}, a_i\| : i = 1, 2, ..., n\Big\}$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, ..., a_n\}$.
A sequence (x_k) in a n-normed space $(X, \|., ..., .\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, .., z_{n-1}\| = 0, \text{ for every } z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) in a normed space $(X, \|., ..., .\|)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \|x_k - x_p, z_1, .., z_{n-1}\| = 0, \text{ for every } z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$ then X is said to be complete with respect to the n -norm. Any complete n-normed space is said to be n-Banach space.

A sequence $x = (x_k) \in l_\infty$, the space of bounded sequence is said to be almost convergent to s if $\lim_{k\to\infty} t_{km}(x) = s$ uniformly in m where $t_{km}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}$ (see [26]).
By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, \cdots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space N_θ of lacunary strongly convergent sequences was defined by Freedman et al. [27] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

By $[\hat{w}]_\theta$, we denote the set of all lacunary $[\hat{w}]$-convergent sequences and we write $[\hat{w}]_\theta - \lim x = s$, for $x \in [\hat{w}]_\theta$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n-X)$ denotes the space of X-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. In this paper, we define the following sequence spaces

$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] =$
$= \Big\{x = (x_k) \in w(n-X) : \frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \to 0$
as $n \to \infty$, uniformly in m, for some $s\Big\}$
and
$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} =$
$\Big\{x = (x_k) \in w(n-X) : \sup_m \frac{1}{h_r}\sum_{k \in I_r} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \to 0$
as $r \to \infty$, for some $s\Big\}$.
The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main goal of the present paper is to introduce $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]_{\theta}$. We also examine some topological properties and prove some inclusion relation between these spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x, y \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}\big(B^{\mu}_{\Lambda}(\alpha x + \beta y) - s\big)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in}$$

m, for some s.

Since $x, y \in [\ \hat{w}(\mathcal{M}, p, \|., \cdots, .\|)\ ]$ there exist positive numbers ρ_1, ρ_2 such that

$$\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for}$$

some s
and

$$\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(y-s))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for}$$

some s.
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M}$ is non-decreasing and convex

$$\begin{aligned}
\frac{1}{n+1} & \sum_{k=0}^{n} M_k\Big(||\frac{t_{km}\big(B_\Lambda^\mu(\alpha x+\beta y)-s\big)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & \frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B_\Lambda^\mu(\alpha x-s))}{\rho_3}, z_1, \cdots, z_{n-1}|| \\
& +||\frac{t_{km}(B_\Lambda^\mu(\beta y-s))}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & \frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B_\Lambda^\mu(x-s))}{\rho_1}, z_1, \cdots, z_{n-1}|| \\
& +||\frac{t_{km}(B_\Lambda^\mu(y-s))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & K\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B_\Lambda^\mu(x-s))}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
+ & K\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B_\Lambda^\mu(y-s))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
& \to 0 \text{ as } n\to\infty, \text{ uniformly in } m, \text{ for some } s.
\end{aligned}$$

Thus $\alpha x+\beta y \in [\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ]$. This proves that $[\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ]$ is a linear space. Similarly, we can prove that $[\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ]_\theta$ is a linear space. □

Theorem 2.2. *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r > 1$. Then $[\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ] \subset [\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ]_\theta$ and $[\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ] - \lim x = [\ \hat{w}(\mathcal{M}, B_\Lambda^\mu, p, \|.,..,.\|)\ ]_\theta - \lim x$.*

Proof. Let $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r > 1+\delta$ and hence

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Therefore,

$$\begin{aligned}
\tfrac{1}{k_r} & \sum_{i=1}^{k_r} M_k\Big(||\frac{t_{im}(B_\Lambda^\mu(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & \frac{1}{k_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B_\Lambda^\mu(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B_\Lambda^\mu(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}
\end{aligned}$$

and if $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x = s$, then it follows that $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = s$. $\square$

Theorem 2.3. *Let* $\theta = (k_r)$ *be a lacunary sequence with* $\limsup q_r < \infty$. *Then* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} \subset [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ *and* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x$.

Proof. Let $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = s$. Then for $\epsilon > 0$, there exists j_0 such that for every $j \geq j_0$ and all m,

$$g_{jm} = \frac{1}{h_j} \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x - s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} < \epsilon,$$

that is, we can find some positive constant C, such that

$$g_{jm} < C \tag{2.1}$$

for all j and m, $\limsup q_r < \infty$ implies that there exists some positive number K such that

$$q_r < K \ \text{ for all } \ r \geq 1. \tag{2.2}$$

Therefore, for $k_{r-1} < n \leq k_r$, we have by (2.1) and (2.2)

$$\begin{aligned}
\frac{1}{n+1} \quad & \sum_{i=0}^{n} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x - s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} \\
\leq \quad & \frac{1}{k_{r-1}} \sum_{i=0}^{k_r} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x - s))}{\rho}, z_1, \cdots z_{n-1} \| \Big)^{p_k} \\
= \quad & \frac{1}{k_{r-1}} \sum_{j=0}^{r} \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x - s))}{\rho}, z_1, z_2, \cdots z_{n-1} \| \Big)^{p_k} \\
= \quad & \frac{1}{k_{r-1}} \Big[\sum_{j=0}^{j_0} \sum_{j=j_0+1}^{r} \Big] \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x - s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} \\
\leq \quad & \frac{1}{k_{r-1}} \Big(\sup_{l \leq p \leq j_0} g_{pm} \Big) k_{j_0} + \epsilon (k_j - k_{j_0}) \frac{1}{k_{r-1}} \\
\leq \quad & C \frac{k_{J_0}}{k_{r-1}} + \epsilon\, K.
\end{aligned}$$

Since $k_{r-1} \to \infty$, we get $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x = s$. This completes the proof of the theorem. $\square$

Theorem 2.4. *Let* $1 < \liminf q_r \leq \limsup q_r < \infty$. *Then* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$.

Proof. It follows directly from Theorem 2.2. and Theorem 2.3. So we omit the details. □

Theorem 2.5. *Let* $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] \cap [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta}$. *Then* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] - \lim x = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x$ *and* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x$ *is unique for any lacunary sequence* $\theta = (k_r)$.

Proof. Let $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] \cap [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta}$. and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x = s$, $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x = s'$. Suppose $s \neq s'$. We see that

$$\begin{aligned} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdot\cdot, z_n||\Big)^{p_k} &\le \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_n||\Big)^{p_k} \\ &+ \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s^{'}))}{\rho}, z_1, \cdots, z_n||\Big)^{p_k} \\ &\le \lim_r \sup_m \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}\Big)^{p_k} + 0. \end{aligned}$$

Hence there exists r_0, such that for $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} > \frac{1}{2}M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}.$$

Since $\Big[\hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), ||., \cdots, .||\Big] - \lim x = s$, it follows that

$$\begin{aligned} 0 &\ge \limsup \frac{h_r}{k_r} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\ &\ge \liminf \frac{h_r}{k_r} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\ &\ge 0 \end{aligned}$$

and so, $\lim q_r = 1$. Hence by Theorem 2.3.,

$$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} \subset [\ \hat{w}(\mathcal{M}, p, \|.,..,.\|)\ ]$$

and

$$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x = s^{'} = s = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] - \lim x.$$

Further,

$$\begin{aligned}
\tfrac{1}{n+1} & \sum_{i=0}^{n} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
+ & \frac{1}{n+1}\sum_{i=0}^{n} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s^{'}))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & 0
\end{aligned}$$

and taking the limit on both sides as $n \to \infty$, we have

$$M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0$$

that is $s = s^{'}$ for Musielak-Orlicz function $\mathcal{M} = (M_k)$ and this completes the proof of the theorem.

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A Sandwich Type Hahn-Banach Theorem for Convex and Concave Functionals

Jingshi Xu

Abstract: We give a sandwich type Hahn-Banach theorem for convex and concave functionals.

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Keywords and Phrases: The Hahn-Banach theorem; Convex functional; Concave functional.

The Hahn-Banach theorem is a fundamental theorem in linear functional analysis. Its sandwich form is the following, see Theorem 3.9 in [5].

Theorem 1 (Sandwich Theorem). *Let $g: X \to \mathbb{R} \cup \{+\infty\}$, and $h: X \to \mathbb{R}$ be sublinear functions on a linear space X. If $-g \leqslant h$, there exists a linear form l on X such that $-g \leqslant l \leqslant h$.*

The following Hahn-Banach extension theorem was given in [1] and [3].

Theorem 2. *Suppose X is a real linear space, p is a convex functional on X, M is a subspace of X. If g is a real linear functional on M such that $g(x) \leqslant p(x),\ x \in M$, then there exists a linear functional f on X such that $f(x) \leqslant p(x),\ \forall x \in X$ and $f(x) = g(x),\ \forall x \in M$.*

In the following we shall use 0 to denote both zero and zero vector. From Theorem 2, we have the following results.

Corollary 1. *Let X be a real linear space and φ be a convex functional on X such that $\varphi(0) \geqslant 0$, then there exists a linear functional L on X such that $L(x) \leqslant \varphi(x)$ for every $x \in X$.*

Proof. Let $E = \{0\}$ and $f_0(0) = 0$, The f_0 is a linear functional on E such that $f_0(x) \leqslant \varphi(x)$ for every $x \in E$. Then by Theorem 2, there exists a linear functional f on X such that $f(x) \leqslant \varphi(x)$ for every $x \in X$. □

Corollary 2. *Suppose that f_0 be a linear functional on subspace M of X, such that $\psi(x) \leqslant f_0(x)$ for every $x \in M$, where ψ is a concave function on X. Then there exists a linear functional L on X such that $L(x) = f_0(x)$ for every $x \in M$ and $\psi(x) \leqslant L(x)$ for every $x \in X$.*

Now, our main result is the following sandwich type theorem for convex and concave functionals.

Theorem 3. *Let M be a subspace in X. Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y \in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. If f_0 is a linear functional on M, then there exists an extension linear functional L on X of f_0 such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$ if and only if $f_0(x) \leqslant T(x)$ for every $x \in M$.*

To give the proof of Theorem 3, we need the following lemmas.

Lemma 1. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y \in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. Let f_0 be a linear functional on a subspace M of X such that*

$$f_0(x) \leqslant T(x) \text{ for every } x \in M. \tag{1}$$

Then the following conditions are satisfied.

(i) *For every $x \in X$, $\psi(x) \leqslant \varphi(x)$;*

(ii) *For every $x \in M$, $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$.*

Proof. From (1), for every $y \in X$ and $x \in M$, $f_0(x) \leqslant \varphi(x+y) - \psi(y)$. Then, let $x = 0$, we have $\psi(y) \leqslant \varphi(y)$ for every $y \in X$. By letting $y = 0$, we see that $f_0(x) \leqslant \varphi(x) - \psi(0) \leqslant \varphi(x)$ for every $x \in M$. By letting $y = -x$, we obtain that $f_0(-y) \leqslant -\psi(y)$ or $\psi(x) \leqslant f_0(x)$ for every $x \in M$. Thus, $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for every $x \in M$. □

Lemma 2. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y \in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. Let $\psi(x) \leqslant \varphi(x)$ for every $x \in X$. Then $\psi(x) \leqslant T(x) \leqslant \varphi(x)$ for every $x \in X$, and T is a convex functional. Moreover, if L is a linear functional on X such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$, then $L(x) \leqslant T(x)$ for every $x \in X$.*

Proof. First, we prove that T is convex. Fix $u, v \in X$. For $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, for every $\epsilon > 0$, there exist $y, z \in X$ such that $\varphi(u+y) - \psi(y) < T(u) + \epsilon$, $\varphi(v+z) - \psi(z) < T(v) + \epsilon$, then

$$\begin{aligned}\varphi(\alpha u + \beta v + \alpha y + \beta z) - \psi(\alpha y + \beta z)\} &\leqslant \alpha\varphi(u+y) + \beta(v+z) - \alpha\psi(y) - \beta\psi(z)\\ &\leqslant \alpha(\varphi(u+y) - \psi(y)) + \beta(\varphi(v+z) - \psi(z))\\ &< \alpha T(u) + \beta T(v) + \epsilon.\end{aligned}$$

Thus $T(\alpha u + \beta v) < \alpha T(u) + \beta T(v) + \epsilon$. Since ϵ is arbitrary, we obtain that $T(\alpha u + \beta v) \leqslant \alpha T(u) + \beta T(v)$. Therefore T is convex.

Since $T(x) \leqslant \varphi(x+y) - \psi(y)$, it follows that $T(x) \leqslant \varphi(x) - \psi(0)$. So $T(x) \leqslant \varphi(x)$ for every $x \in X$. Again, $T(-y) \leqslant \varphi(0) - \psi(y)$, So $T(-y) \leqslant -\psi(y)$. Since $T(0) = 0$, and by the convexity of T, $0 \leqslant T(0) \leqslant 1/2T(y) + 1/2T(-y)$, so that $-T(y) \leqslant T(-y)$. Hence, $-T(y) \leqslant T(-y) \leqslant -\psi(y)$. Thus $\psi(y) \leqslant T(y)$ for every $y \in X$. Consequently, $\psi(x) \leqslant T(x) \leqslant \varphi(x)$ for every $x \in X$.

Finally, suppose that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$. Now $\psi(u) \leqslant L(u)$, it follows that $L(u) \leqslant -\psi(-u)$. Hence, by the linearity of L we obtain that $L(u+v) \leqslant \varphi(v) - \psi(-u)$ for every $u, v \in X$. Letting $v = x + y$ and $u = -y$, we obtain $L(x) \leqslant \varphi(x+y) - \psi(y)$. Taking the infimum over all $y \in X$, we obtain that $L(x) \leqslant T(x)$ for every $x \in X$. □

Proof of Theorem 3. If a linear functional L on X is an extension of f_0 such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$. By Lemma 2, $L(x) \leqslant T(x)$ for every $x \in X$. Since $f_0(x) = L(x)$ for each $x \in M$, so $f_0(x) \leqslant T(x)$ for every $x \in M$.

Conversely, if $f_0(x) \leqslant T(x)$ for every $x \in M$, by Lemma 1, $\psi(x) \leqslant \varphi(x)$ for all $x \in X$ and $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for all $x \in M$. According to Lemma 2, we see that T is a convex function. Now, by Theorem 2 there is an extension linear functional L on X such that $f_0(x) = L(x)$ for each $x \in M$ and $L(x) \leqslant T(x)$ for each $x \in X$. By Lemma 1, $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for all $x \in X$. □

By Theorem 3, we have an generalization of Theorem 1 as follows.

Theorem 4. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y \in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. If $\psi(x) \leqslant \varphi(x)$ for every $x \in X$, then there exists a linear functional L on X such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$.*

Proof. Let $E = \{0\}$ and $f_0(0) = 0$, The f_0 is a linear functional on E such that $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for every $x \in E$. Then by Theorem 3, there exists a linear functional f on X such that $\psi(x) \leqslant f(x) \leqslant \varphi(x)$ for every $x \in X$. □

In Theorems 3 and 4, the condition $\varphi(0) = \psi(0) = 0$ is necessary. For example, in $\mathbb{R}$, let $\varphi(x) = (x+4)^2 - 4$, $\psi(x) = -e^x - 4$, then there exists no constant k such that $\varphi(x) \geqslant kx \geqslant \psi(x)$ for all $x \in \mathbb{R}$.

Remark 1. Theorem 4 partly generalises the sandwich version Hahn-Banach Theorem in [2]. Páles gave a different type Sandwich theorems in [4].

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