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A Characterization of Weakly J(n)-Rings

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Abstract: A ring R is called a *J(n)-ring* if there exists a natural number $n \geq 1$ such that for each element $r \in R$ the equality $r^{n+1} = r$ holds and a *weakly J(n)-ring* if there exists a natural number $n \geq 1$ such that for each element $r \in R$ the equalities $r^{n+1} = r$ or $r^{n+1} = -r$ hold.

We completely describe both classes of these rings R for any n, thus considerably extending some well-known results in the subject, especially that of V. Perić in Publ. Inst. Math. Beograd (1983) as well as, in particular, the classical description of Boolean rings when $n = 1$.

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1. Introduction and Background Material

Throughout, all rings R examined in the current paper shall be assumed associative, containing the identity element 1 which possibly differs from the zero element 0. Standardly, $U(R)$ denotes the set of all invertible elements of R, $Id(R)$ the set of all idempotent elements of R and $Nil(R)$ the set of all nilpotent elements of R. Traditionally, $J(R)$ denotes the Jacobson radical of R. All other notions and notations, not explicitly defined herein, are well-established in the existing literature. About the specific terminology, specifically that of a *PI-ring*, let us recall that it is a ring whose elements satisfy a polynomial identity with coefficients in $\mathbb{Z}$, the ring of all integers, and at least one coefficient has to be invertible, that is, ± 1. In particular, commutative rings are always PI-rings.

The following concept is rather well-known.

Definition 1.1. A ring R is called *Boolean* if $x^2 = x$ for each $x \in R$, that is, $R = Id(R)$.

These rings have a complete characterization as (the subring of) the direct product of family of copies of the two element field $\mathbb{Z}_2$ (see, e.g., [2]). Hence, Boolean rings are themselves commutative.

To consider some other generalizations, for a fixed prime p, a *p-ring* is a ring R in which $a^p = a$ and $pa = 0$ for all $a \in R$. Thus any Boolean ring is simply a 2-ring. It is known that a ring is a p-ring if, and only if, it is a subdirect product of fields of order p (cf. [15]). On the other hand, for a prime p and a positive integer k, a *p^k-ring* is a ring R in which $a^{p^k} = a$ and $pa = 0$ for all $a \in R$. The structure of p^k-rings has been described in [1]. A ring R is said to be *periodic* if, for each $a \in R$, there is a positive integer $n(a)$ such that $a^{n(a)+1} = a$. Every periodic ring is commutative by a fundamental result due to Jacobson from [12]. If such a natural number $n(a)$ does not depend on the choice of the element a, that is it could be fixed, we thus come to the following concept.

Definition 1.2. Let $n \geq 1$ be a natural number. We shall say that the ring R is a *J(n)-ring* if, for every $x \in R$, the equation $x^{n+1} = x$ holds.

The special case when $n = 1$ gives the famous Boolean rings; notice also that $x = x^2$ always implies that $x = x^j$ for all $j \in \mathbb{N}$. Likewise, these rings are obviously perfect, i.e., $R = R^{n+1}$. Moreover, it was proved in [14] that a ring is a J(n)-ring if, and only if, it is the direct sum of finitely many p^k-rings. Hence, with the aforementioned result in [1] at hand, the structural characterization of J(n)-rings can be assumed for totally exhausted. On the other side, they are also somewhat studied in [10], but without any concrete full description given.

However, the complete description of J(n)-rings was given in [17]. There was proved that *R is a J(n)-ring if, and only if, R is a subdirect product of fields $\mathbb{F}_{p^k}$, where p is a prime and k is an integer such that $p^k - 1$ divides n.* Nevertheless, we will give here a new more convenient and attractive for further applications description of their structure in terms of the simple p-element fields $\mathbb{Z}_p$, where p is a prime, and the fields $\mathbb{F}_q$ of $q = p^k$ elements, where $k \in \mathbb{N}$. So, the objective of this article is to do that by using an elementary algebraic approach. We refer also to [3] and [13] for more account to that topic.

In order to substantially enlarge the above explorations, we shall be concerned here and with giving up the full characterization up to isomorphism of weakly J(n)-rings, that are, rings whose elements satisfy the polynomial identities $x^{n+1} - x = 0$ or $x^{n+1} + x = 0$.

We thus come to the following new concept:

Definition 1.3. Suppose that $n \geq 1$ be a natural number. We shall say that the ring R is a *weakly J(n)-ring* if, for every $x \in R$, the equations $x^{n+1} = x$ or $x^{n+1} = -x$ hold.

It is pretty obvious that subrings and homomorphic images of (weakly) J(n)-rings are again (weakly) J(n)-rings. Some concrete folklore examples are these:

• "$n = 1$": *A ring R is a J(2)-ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

As usual, $\mathbb{F}_4$ denotes the field of characteristic 2 consisting of four elements constructed as follows: It is well known that in the polynomial ring $\mathbb{Z}_2[x]$ the polynomial $1 + x + x^2$ is irreducible over $\mathbb{Z}_2$ and hence $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{Z}_2(\theta)$ is a field of 4 elements, denoting it by $\mathbb{F}_4$, where $\theta \notin \mathbb{Z}_2$ is a solution of the equation $x^3 = 1$. In fact, the elements in $\mathbb{F}_4$ are $\{0, 1, \theta, \theta^2\}$ taking into account that $\theta + 1 = \theta^{-1} = \theta^2$ and that $x^3 - 1 = (x^2 + x + 1)(x - 1)$ whence $x^3 = 1$ has the set $\{1 = \theta^0, \theta, \theta^2\}$ as solutions. Thus $\mathbb{F}_4$ is the splitting field of these two polynomials. Note that under such a construction the equality $\mathbb{F}_2 = \mathbb{Z}_2$ is true.

• "$n = 2$": *A ring R is a $J(3)$-ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{F}_4$.*

• "$n = 3$": *A ring R is a $J(4)$-ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$.*

In what follows in the subsequent section, we shall provide a full characterizing of both J(n)-rings and weakly J(n)-rings.

2. The Main Results

We start in this section with the following technicality by treating the general case of J(n)-rings, as our purpose is to give a new more transparent and conceptual proof of the characterization result for these rings than that in [17]. In doing that, we need the following technical claim.

Lemma 2.1. *Let $n \in \mathbb{N}$ and let R be a ring whose elements satisfy the identity $x^{n+1} = x$, while $x^{k+1} \neq x$ for some x, provided $k < n$ and $k \in \mathbb{N}$, that is, for every $k < n$ there exists x in R for which x^{k+1} is not equal to x. The next three items are true:*

1. *R is reduced.*
2. *$J(R) = 0$.*
3. *If R is primitive, then $n = p^m - 1$ for some $m \in \mathbb{N}$ and R is a field with p^m elements.*

Proof. Items (1) and (2) are rather obvious, which follow directly from the condition $x^{n+1} = x$, so we omit their verification. The third item is an immediate consequence of the fact that R is a PI-ring and of the well-known Kaplansky's theorem by using the method presented in detail in [8]. □

We are now proceed by proving with the following basic statement, which somewhat improves on the aforementioned characterizing result from [17] concerning J(n)-rings.

Theorem 2.2. *Suppose that $n \in \mathbb{N}$. Then, for a ring R, the following two conditions are equivalent:*

1. *R is a J(n)-ring.*
2. *R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k, $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k.*

Proof. "(1) $\Rightarrow$ (2)". With Lemma 2.1 at hand, R is a subdirect products of finite fields F_i satisfying the equality $x^{n+1} = x$. Let us fix such a field F with p^m elements. It is then well known that $U(F)$ is a cyclic group of order $p^m - 1$ which satisfies the identity $x^n = 1$. Thus $p^m - 1$ divides n.

"(2) $\Rightarrow$ (1)". Letting R be a subdirect product of the fields F_i, we then easily see that each field satisfies $x^{n+1} = x$, thus R will also satisfy this identity. □

By the same token, we can derive the following consequence for J(n)-rings in the presence of minimality of the existing natural number n.

Corollary 2.3. *Suppose $n \in \mathbb{N}$. Then, for a ring R, the following two conditions are tantamount:*

1. *R satisfies the equation $x^{n+1} = x$ with n minimal possible.*
2. *R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k, $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k and $n = LCM(p_k^{m_k} - 1 \mid k \in \mathbb{N})$.*

For a convenience of the reader, let us recall that $\mathbb{F}_q$ is the finite field with q elements with q a prime power. We are now ready to prepare our chief result which completely settles the question when an arbitrary ring is weakly J(n) and which states as follows:

Theorem 2.4. *Let $n \geq 1$ be a natural and let R be a ring. Then R is a weakly J(n)-ring if, and only if, R is a J(n)-ring, or either $R = \mathbb{F}_q$ or $R = P \times \mathbb{F}_q$, where $q - 1$ divides $2n$ but not n, and P is a J(n)-ring of characteristic 2.*

Proof. In one direction, if $R = P \times \mathbb{F}_q$, where P is the zero ring or a J(n)-ring of characteristic 2, for any pair $(x, y) \in R$ we indeed have $(x, y)^{n+1} = \pm(x, y)$ depending on the fact whether $y \in \mathbb{F}_q$ is a square or not.

In the other direction, we first observe that $x^{2n+1} = x$ for all $x \in R$. By the famous Jacobson's theorem for commutativity, R is really commutative. Moreover, by the usage of classical arguments, R is the direct product of characteristic p rings, where p is one of the finitely many primes, for which $p - 1$ divides $2n$. Then, at least one of these rings, say K, contains an element a with $a^{n+1} \neq a$. Certainly, $\mathrm{char}(K) = p > 2$. If foremost $R \neq K$, then R is of the form $K \times P$, where P is a non-zero ring. Consider the element $(a, 1)$ in R. Its $n+1$-th power is $(a^{n+1}, 1)$, and is equal to $\pm(a, 1)$. Since a^{n+1} differs from a, it must be that $(a^{n+1}, 1) = -(a, 1) = (-a, -1)$ whence $1 = -1$ in P and, therefore, it follows that P is necessarily of characteristic 2.

So, it suffices to show that $K = \mathbb{F}_q$, where q has the properties indicated in the statement of the theorem. To that goal, since the element a satisfied the inequality $a^{n+1} \neq a$, we must have $a^{n+1} = -a$. Besides, since every prime ideal of K is maximal and since the nil-radical of K is trivial, there exists a maximal ideal I of K for which $a^{n+1} = -a$ modulo I. Then the field K/I is finite of characteristic p. We denote it by $\mathbb{F}_q$. Now, every element $y \in K/I$ satisfies $y^{2n+1} = y$. Since $\mathbb{F}_q^*$ is a cyclic group of order $q-1$, $q-1$ divides $2n$. Since $a^{n+1} = -a$ and the characteristic of $\mathbb{F}_q$ is not 2, the number n is not divisible by $q-1$.

If now K admits a second maximal ideal M, then I and M are co-prime, so that $(K/I) \times (K/M)$ is a quotient of K. Consider its element $(a, 1)$. Then $(a, 1)^{n+1} = (-a, 1)$ which is not equal to $\pm(a, 1)$ – a contradiction. Thus I is the only maximal ideal of K. This means that K is a local ring, and hence equal to its residue field $K/I = \mathbb{F}_q$. The last claim is a direct consequence of the fact that $x^{n+1} = \pm x$ for all $x \in K$. □

Remark 2.5. It is worthwhile noticing that the cases $n = 1$ are settled in [9] and [4]; $n = 2$ in [5]; and $n = 3$ in [6]. Moreover, by virtue of the main Theorem 2.4, in accordance with Theorem 2.2, or with the main result from [17], the study of the structure of weakly J(n)-rings is completely exhausted.

We shall now be involved with some applications by giving up a slight generalization of J(n)-rings for $n \in \mathbb{N}$ to rings with elements satisfying the equation $x^n = \varsigma x$, where ς is a (primitive) d-th root of unity for some positive integer d (compare with the slightly weaker version stated in Problem 3 listed below). Precisely, the following assertion is true:

Theorem 2.6. *Let R be a commutative ring and let $f(X) = \sum_{n\geq 0} a_n X^n$ be a polynomial in $R[X]$. Suppose also that the polynomial $\widetilde{f}(X) = \sum_{n\geq 0} a_n a_0^{n-1} X^n$ has a root in R and that $a_n a_0^{n-1} = 1$ for $n = 0$. Then, for all $y \in R$ and each ideal $I \subseteq Ann(y)$, the annihilator of y, we have*

$$f(y) \equiv 0 (mod\ I) \iff \exists\, \varsigma \in R\ :\ \varsigma \equiv y (mod\ I) \bigwedge f(\varsigma) = 0.$$

Proof. The implication "⇐" being elementary, we will be concentrated on the reverse one "⇒". To that goal, let $\lambda \in R$ be a zero of $\widetilde{f}(X)$. We therefore readily check that $\varsigma = y + \lambda f(y)$ is a zero of $f(X)$. Keeping in mind that $yf(y) = 0$, we deduce that

$$f(y+\lambda f(y)) = \sum_{n\geq 0} a_n (y+\lambda f(y))^n = f(y) + \sum_{n\geq 1} a_n \lambda^n f(y)^n = f(y) + \sum_{n\geq 1} a_n \lambda^n f(y) a_0^{n-1}.$$

This is obviously equal to

$$f(y) \sum_{n\geq 0} a_n a_0^{n-1} \lambda^n = f(y) \widetilde{f}(\lambda) = 0,$$

as required. □

As a valuable consequence, one derives the following.

Corollary 2.7. *Let R be a ring and let d, n be positive integers. Then, for all $x \in R$, we have*

$$x^{d(n-1)+1} = x \iff \exists\, \varsigma \in R \;:\; \varsigma^d = 1 \bigwedge x^n = \varsigma x.$$

Proof. Let $x \in R$. Since for any $\varsigma \in R$ and for $i = d, \ldots, 1$, we obtain

$$x^{i(n-1)+1} = \varsigma x x^{i(n-1)+1-n} = \varsigma x^{(i-1)(n-1)+1},$$

the implication "$\Leftarrow$" is clear. To prove the converse implication "$\Rightarrow$", we may with no harm in generality replace R by the subring generated by 1 and x. In particular, we can assume even that R is commutative.

Now, put $I = Ann(x)$. Consequently, the statement of the corollary is equivalent to

$$x^{d(n-1)} \equiv 1(mod\ I) \iff \exists\, \varsigma \in R \;:\; \varsigma^d = 1 \bigwedge \varsigma \equiv x^{n-1}(mod\ I).$$

Next put $f(X) = X^d - 1 \in R[X]$ and $y = x^{n-1}$; thus $f(\varsigma) = 0$. So, the statement of the corollary is amounting to

$$f(y) \equiv 0(mod\ I) \iff \exists\, \varsigma \in R \;:\; f(\varsigma) = 0 \bigwedge \varsigma \equiv y(mod\ I).$$

Note that the ideal I is pretty obviously contained in $\mathrm{Ann}(y)$. Since $\lambda = -1$ is a zero of $\widetilde{f}(X) = (-1)^{d-1}X^d + 1$, the condition of the previous theorem is satisfied and thus the corollary follows after all. □

It is worthwhile noticing that Theorem 2.6 perhaps can be considerably extended in the non-commutative case as follows (actually, its formulation smells a little like the classical well-known Hensel's lemma):

Let R be a ring and let $f(X) \in R[X]$ be a polynomial with coefficients in the center of R. Suppose also that $f(X)$ has an invertible root in the center of R and that $f(0)$ inverts in R. Then, for all $y \in R$ and every ideal $I \subset Ann(y)$, the annihilator of y, we have

$$f(y) \equiv 0(mod\ I) \iff \exists\, \varsigma \in R \;:\; \varsigma \equiv y(mod\ I) \bigwedge f(\varsigma) = 0.$$

Some idea for an eventual proof could be the following: The right-to-left part being self-evident, we will deal with the left-to-right one. To that purpose, let $\varepsilon \in R$ be an invertible central root of $f(X)$. Put

$$h(X) = \frac{\varepsilon f(X)}{f(0)}.$$

Notice that $h(0) = \varepsilon$ inverts in R. Thus $h(y)$ lies in I. We easily check that $\varsigma = y+h(y)$ satisfies the equality $f(\varsigma) = 0$. Writing $f(X) = \sum_{n\geq 0} a_n X^n$ and bearing in mind that $yh(y) = 0$, one infers that

$$f(y + h(y)) = \sum_{n\geq 0} a_n(y + h(y))^n = f(y) + \sum_{n\geq 1} a_n h(y)^n.$$

We however see that $h(y)^n = h(0)^{n-1}h(y)$ for every $n \geq 1$. Therefore, we get that

$$0 = f(y + h(y)) = f(y) + h(y) \sum_{n\geq 1} a_n h(0)^{n-1} = f(y) - \frac{h(y)}{h(0)} f(0) + \frac{h(y)}{h(0)} f(h(0)).$$

Since $h(0) = \varepsilon$ and $f(\varepsilon) = 0$, the last expression on the right must be zero, and thus $f(y) = \frac{h(y)}{h(0)} f(0)$ is in I, which demonstrably riches us that we are done.

3. Concluding Discussion and Open Questions

We call a ring R *π-simply presented* if, for any $a \in R$, there exists an integer $n = n(a) \geq 2$ such that either $a^n = a$ or $a^n = 0$. If such a natural n is fixed, and so it does not depend on a, the ring R is just called *n-simply presented*.

It is rather clear that 2-simply presented rings R are just the Boolean ones. In fact, if $u \in U(R)$, then $u^2 = u$ and hence $u = 1$. This means that $U(R) = \{1\}$ whence $Nil(R) = \{0\}$. Consequently, for every $r \in R$, it must be that $r^2 = r$, as required. It is worthwhile noticing that this could also be deduced from [7].

Recall also that (see, e.g., [16]) a ring is *strongly clean* if every its element is the sum of a unit and an idempotent which commute each to other. Thus the following is true: *Any n-simply presented ring is strongly clean with nil Jacobson radical.* In fact, if R is such a ring and $a \in R$ with $a^n = 0$, one represents $a = (a-1)+1 \in U(R)+Id(R)$. If now $a^n = a$, one checks that $a = a^2 b = ba^2$, where $b = a^{n-1} + a^{n-2} - 1$. Therefore, a is strongly regular element and, thereby, it follows from [16] that such an element a is strongly clean. Finally, this enables us that R is strongly clean. Also, it is pretty easy to find that $U(R)$ is torsion having $U^{n-1}(R) = \{1\}$ and thus, in view of [8], one infers that R is strongly m-torsion clean for some $m \leq n$. We, consequently, again appeal to [8] to get some expected subdirect isomorphism.

About the nil property of $J(R)$, choose an arbitrary $z \in J(R)$. If $z^n = 0$, we are set. If, however, $z^n = z$, then $z(1 - z^{n-1}) = 0$ which allows us to conclude that $z = 0$ since $1 - z^{n-1} \in U(R)$.

We close the work with the following three problems of interest and importance.

Problem 1. Describe n-simply presented rings for all $n \in \mathbb{N}$ as well as π-simply presented rings.

Problem 2. For an arbitrary natural $n \geq 1$ and a ring R such that its characteristic is the prime number $n+1$, does it follow that R is a J(n)-ring if, and only if, each element of R is a sum of n idempotents? Equivalently, is such a ring R a J(n)-ring exactly when $R = Id(R) + \cdots + Id(R)$ (where the sum is taken n-times)?

We conjecture that the answer is "**yes**", provided that $n+1$ runs over some special primes by noticing in this way that if $n = 1$, we just identify Boolean rings, and that if $n = 2$, the conjecture holds in the affirmative in accordance with [11, Theorem 1].

Problem 3. Describe those rings whose elements satisfy the polynomial identity $x^{n+1} - vx = 0$, where $n \in \mathbb{N}$ and $v^2 = 1$.

It is pretty obvious that weakly J(n)-rings are a partial case of these rings, when $v = \pm 1$.

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