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Existence and Uniqueness of Solutions for Nonlinear Katugampola Fractional Differential Equations

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Abstract: The present paper deals with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations with Katugampola fractional derivative. The main results are proved by means of Guo-Krasnoselskii and Banach fixed point theorems. For applications purposes, some examples are provided to demonstrate the usefulness of our main results.

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1. Introduction

The differential equations of fractional order are generalizations of classical differential equations of integer order. They are increasingly used in a variety of fields such as fluid flow, control theory of dynamical systems, signal and image processing, aerodynamics, electromagnetics, probability and statistics, (Samko et al. 1993 [18], Podlubny 1999 [17], Kilbas et al. 2006 [9], Diethelm 2010 [3]) books can be checked as a reference.

Boundary value problem of fractional differential equations is recently approached by various researchers ([1], [8], [19], [20]).

In [20], Bai and L used some fixed point theorems on cone to show the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation:

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

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where $\mathcal{D}_{0+}^{\alpha}u$ is the standard Riemann Liouville fractional derivative of order $1 < \alpha \leq 2$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous function.

In a recent work [8], Katugampola studied the existence and uniqueness of solutions for the following initial value problem:

$$\begin{cases} {}_{c}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) = f(t, u(t)), \ \alpha > 0, \\ D^{k}u(0) = u_{0}^{(k)}, \ k = 1, 2, ..., m-1, \end{cases}$$

where $m = [\alpha]$, ${}_{c}^{\rho}\mathcal{D}_{0+}^{\alpha}$ is the Caputo-type generalized fractional derivative, of order α, and $f : G \to \mathbb{R}$ is a given continuous function with:

$$G = \left\{ (t, u) : t \in [0, h^{*}], \ \left| u - \sum_{k=0}^{m-1} \frac{t^{k} u_{0}^{(k)}}{k!} \right| \leq K, \ K, h^{*} > 0 \right\}.$$

This paper focuses on the existence and uniqueness of solutions for a nonlinear fractional differential equation involving Katugampola fractional derivative:

$$^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) + \beta f(t, u(t)) = 0, \ 0 < t < T, \tag{1.1}$$

supplemented with the boundary conditions:

$$u(0) = 0, \ u(T) = 0, \tag{1.2}$$

where $\beta \in \mathbb{R}$, and $^{\rho}\mathcal{D}_{0+}^{\alpha}$ for $\rho > 0$, presents Katugampola fractional derivative of order $1 < \alpha \leq 2$, $f : [0, T] \times [0, \infty) \to [h, \infty)$ is a continuous function, with finite positive constants h, T.

2. Background materials and preliminaries

In this section, some necessary definitions from fractional calculus theory are presented. Let $\Omega = [0, T] \subset \mathbb{R}$ be a finite interval.

As in [9], let us denote by $X_{c}^{p}[0, T]$, $(c \in \mathbb{R}, \ 1 \leq p \leq \infty)$ the space of those complex-valued Lebesgue measurable functions y on $[0, T]$ for which $\|y\|_{X_{c}^{p}} < \infty$ is defined by

$$\|y\|_{X_{c}^{p}} = \left(\int_{0}^{T} |s^{c} y(s)|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$, and

$$\|y\|_{X_{c}^{\infty}} = \text{ess} \sup_{0 \leq t \leq T} [t^{c} |y(t)|], \ (c \in \mathbb{R}).$$

Definition 2.1 (Riemann-Liouville fractional integral [9])**.** The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{RL}\mathcal{I}_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} y(s) \, ds, \ t \in [0, T],$$

where $\Gamma(\alpha) = \int_{0}^{+\infty} e^{-s} s^{\alpha-1} ds$, is the Euler gamma function.

Definition 2.2 (Riemann-Liouville fractional derivative [9])**.** The left-sided Riemann Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{RL}\mathcal{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds, \ t \in [0, T], \ n = [\alpha] + 1,$$

Definition 2.3 (Hadamard fractional integral [9])**.** The left-sided Hadamard fractional integral of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{H}\mathcal{I}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}, \quad t \in [0, T].$$

Definition 2.4 (Hadamard fractional derivative [9])**.** The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{H}\mathcal{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^{n} \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{ds}{s}, \quad t \in [0, T], \ n = [\alpha] + 1,$$

if the integral exist.

A recent generalization in 2011, introduced by Udita Katugampola [6], combines the Riemann-Liouville fractional integral and the Hadamard fractional integral into a single form (see [9]) , the integral is now known as Katugampola fractional integral, it is given in the following definition:

Definition 2.5 (Katugampola fractional integral [6])**.**
The left-sided Katugampola fractional integral of order $\alpha > 0$ of a function $y \in X_c^p[0, T]$ is defined by:

$$\left({}^{\rho}\mathcal{I}_{0+}^{\alpha} y\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} y(s)}{(t^{\rho} - s^{\rho})^{1-\alpha}} ds, \ \rho > 0, \ t \in [0, T]. \tag{2.1}$$

Similarly, we can define right-sided integrals [6]-[7], [9].

Definition 2.6 (Katugampola fractional derivatives [7])**.**
Let $\alpha, \rho \in \mathbb{R}^{+}$, and $n = [\alpha] + 1$. The Katugampola fractional derivative corresponding to the Katugampola fractional integral (2.1) are defined for $0 \leq t \leq T \leq \infty$ by:

$$^{\rho}\mathcal{D}_{0+}^{\alpha} y(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^{n} \left({}^{\rho}\mathcal{I}_{0+}^{n-\alpha} y\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^{n} \int_0^t \frac{s^{\rho-1} y(s)}{(t^{\rho} - s^{\rho})^{\alpha-n+1}} ds. \tag{2.2}$$

Theorem 2.7 ([7]). *Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$\begin{aligned}
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{I}^{\alpha}_{0^+}y\right)(t) &= {}^{RL}\mathcal{I}^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{I}^{\alpha}_{0^+}y\right)(t) &= {}^{H}\mathcal{I}^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds,\\
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{D}^{\alpha}_{0^+}y\right)(t) &= {}^{RL}\mathcal{D}^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}y(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{D}^{\alpha}_{0^+}y\right)(t) &= {}^{H}\mathcal{D}^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_0^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{y(s)}{s}ds.
\end{aligned}$$

Remark. As an example, for $\alpha, \rho > 0$, and $\mu > -\rho$, we have

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}t^{\mu-\alpha\rho}. \tag{2.3}$$

In particular

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\rho(\alpha-m)} = 0, \text{ for each } m = 1,2,\ldots,n.$$

For $\mu > -\rho$, we have

$$\begin{aligned}
{}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\mu} &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho+\mu-1}\left(t^{\rho}-s^{\rho}\right)^{n-\alpha-1}ds\\
&= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}\int_0^1 \tau^{\frac{\mu}{\rho}}(1-\tau)^{n-\alpha-1}d\tau\\
&= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)}B\left(n-\alpha, 1+\frac{\mu}{\rho}\right)\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}\\
&= \frac{\rho^{\alpha-n}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right)}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}.
\end{aligned}$$

Then

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right)}\left[n-\alpha+\frac{\mu}{\rho}\right]\left[n-\alpha-1+\frac{\mu}{\rho}\right]\cdots\left[1-\alpha+\frac{\mu}{\rho}\right]t^{\mu-\alpha\rho}. \tag{2.4}$$

As

$$\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right) = \left[n-\alpha+\frac{\mu}{\rho}\right]\left[n-\alpha-1+\frac{\mu}{\rho}\right]\cdots\left[1-\alpha+\frac{\mu}{\rho}\right]\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right),$$

we get

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}t^{\mu-\alpha\rho}.$$

In case $m = \alpha - \frac{\mu}{\rho}$, it follows from (2.4), that

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = \rho^{\alpha-1} \frac{\Gamma(\alpha-m+1)}{\Gamma(n-m+1)} (n-m)(n-m-1)\cdots(1-m)\, t^{-\rho m}.$$

So, for $m = 1, 2, \ldots, n$, we get

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0.$$

Similarly, for all $\alpha, \rho > 0$, we have:

$$^{\rho}\mathcal{I}_{0+}^{\alpha} t^{\mu} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)} t^{\mu+\alpha\rho}, \ \forall \mu > -\rho. \tag{2.5}$$

By $C[0,T]$, we denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm:

$$\|y\| = \max_{0\leq t\leq T} |y(t)|.$$

Remark. Let $p \geq 1$, $c > 0$ and $T \leq (pc)^{\frac{1}{pc}}$. Far all $y \in C[0,T]$, note that

$$\|y\|_{X_c^p} = \left(\int_0^T |s^c y(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} \leq \left(\|y\|^p \int_0^T s^{pc-1} ds\right)^{\frac{1}{p}} = \frac{T^c}{(pc)^{\frac{1}{p}}} \|y\|,$$

and

$$\|y\|_{X_c^\infty} = \text{ess} \sup_{0\leq t\leq T} [t^c |y(t)|] \leq T^c \|y\|,$$

which imply that $C[0,T] \hookrightarrow X_c^p[0,T]$, and

$$\|y\|_{X_c^p} \leq \|y\|_\infty, \text{ for all } \ T \leq (pc)^{\frac{1}{pc}}.$$

We express some properties of Katugampola fractional integral and derivative in the following result.

Theorem 2.8 ([6]-[7]-[8])**.**
Let $\alpha, \beta, \rho, c \in \mathbb{R}$, be such that $\alpha, \beta, \rho > 0$. Then, for any $y \in X_c^p[0,T]$, where $1 \leq p \leq \infty$, we have:
- Index property:

$$\begin{aligned} ^{\rho}\mathcal{I}_{0+}^{\alpha}\, {}^{\rho}\mathcal{I}_{0+}^{\beta} y(t) &= {}^{\rho}\mathcal{I}_{0+}^{\alpha+\beta} y(t), \quad \textit{for all } \alpha, \beta > 0, \\ ^{\rho}\mathcal{D}_{0+}^{\alpha}\, {}^{\rho}\mathcal{D}_{0+}^{\beta} y(t) &= {}^{\rho}\mathcal{D}_{0+}^{\alpha+\beta} y(t), \quad \textit{for all } 0 < \alpha, \beta < 1. \end{aligned}$$

- Inverse property

$$^{\rho}\mathcal{D}_{0+}^{\alpha}\, {}^{\rho}\mathcal{I}_{0+}^{\alpha} y(t) = y(t), \quad \textit{for all } \alpha \in (0,1).$$

From Definitions 2.5 and 2.6, and Theorem 2.8, we deduce that

$$
\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right) &= \int_0^t s^{\rho-1}\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(s\right)ds \\
&= \int_0^t \frac{d}{ds}\,{}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(s\right)ds \\
&= \left[\frac{1}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}\int_0^s \tau^{\rho-1}\left(t^{\rho}-\tau^{\rho}\right)^{\alpha}y\left(\tau\right)d\tau\right]_0^t \\
&= {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right).
\end{aligned}
$$

Consequently

$$\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right) = {}^{\rho}\mathcal{I}_{0^+}^{\alpha}y\left(t\right), \ \forall\alpha > 0. \tag{2.6}$$

Definition 2.9 ([4]). Let E be a real Banach space, a nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following conditions:
(i) $u \in P$, $\lambda \geq 0$, implies $\lambda u \in P$.
(ii) $u \in P$, $-u \in P$, implies $u = 0$.

Definition 2.10 ([2]). Let E be a Banach space, $P \in C\left(E\right)$ is called an equicontinuous part if and only if

$$\forall\varepsilon > 0,\ \exists\delta > 0,\ \forall u, v \in E,\ \forall\mathcal{A} \in P,\ \left\|u - v\right\| < \delta \Rightarrow \left\|\mathcal{A}\left(u\right) - \mathcal{A}\left(v\right)\right\| < \varepsilon.$$

Theorem 2.11 (Ascoli-Arzel [2]). *Let E be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C\left(E\right)$, then $\mathcal{A}$ is relatively compact.*

Definition 2.12 (Completely continuous [4]). We say $\mathcal{A} : E \to E$ is completely continuous if for any bounded subset $P \subset E$, the set $\mathcal{A}\left(P\right)$ is relatively compact.

The following fixed-point theorems are fundamental in the proofs of our main results.

Lemma 2.13 (Guo-Krasnosel'skii fixed point theorems [12]).
Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $\mathcal{A} : P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to P$ is a completely continuous operator such that either

(i) $\left\|\mathcal{A}x\right\| \leq \left\|x\right\|$, $x \in P \cap \partial\Omega_1$ *and* $\left\|\mathcal{A}x\right\| \geq \left\|x\right\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\left\|\mathcal{A}x\right\| \geq \left\|x\right\|$, $x \in P \cap \partial\Omega_1$ *and* $\left\|\mathcal{A}x\right\| \leq \left\|x\right\|$, $x \in P \cap \partial\Omega_2$,

holds. Then $\mathcal{A}$ has a fixed point in $P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

Theorem 2.14 (Banach's fixed point [5]). *Let E be a Banach space, $P \subseteq E$ a nonempty closed subset. If $\mathcal{A} : P \to P$ is a contraction mapping, then $\mathcal{A}$ has a unique fixed point in P.*

3. Main results

In the sequel, T, p and c are real constants such that

$$p \geq 1, \ c > 0, \text{ and } T \leq (pc)^{\frac{1}{pc}}.$$

Now, we present some important lemmas which play a key role in the proofs of the main results.

Lemma 3.1. *Let $\alpha, \rho \in \mathbb{R}^+$. If $u \in C[0,T]$, then:*

(i) *The fractional equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, has a solution as follows:*

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \ \text{where } C_m \in \mathbb{R}, \ \text{with } m = 1, 2, \ldots, n.$$

(ii) *If ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ and $1 < \alpha \leq 2$, then:*

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}\ u(t) = u(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}, \quad \text{for some } C_1, C_2 \in \mathbb{R}. \quad (3.1)$$

Proof. (i) Let $\alpha, \rho \in \mathbb{R}^+$. From remark 2, we have:

$${}^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0, \quad \text{for each } m = 1, 2, \ldots, n.$$

Then, the fractional differential equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, admits a solution as follows:

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \ C_m \in \mathbb{R}, \ m = 1, 2, \ldots, n.$$

(ii) Let ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ be the fractional derivative (2.2) of order $1 < \alpha \leq 2$. If we apply the operator ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ to ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t)$ and use Definitions 2.5, 2.6, Theorem 2.8 and property (2.6), we get

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) &= \left(t^{1-\rho}\frac{d}{dt}\right) {}^{\rho}\mathcal{I}_{0+}^{\alpha+1}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha} s^{\rho-1}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\, ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha} s^{\rho-1}\left[\left(s^{1-\rho}\frac{d}{ds}\right)^2 {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right] ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha} \frac{d}{ds}\left[\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right] ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(\left[(t^{\rho}-s^{\rho})^{\alpha}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right]_0^t\right.\right. \\
&\quad \left.\left. + \ \alpha\rho\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\, ds\right)\right].
\end{aligned}$$

From (2.6) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)\,=\,{}^{\rho}\mathcal{I}_{0^{+}}^{1-\alpha}u\left(s\right). \tag{3.2}$$

On the other hand, from (2.2) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)\,=\left(s^{1-\rho}\frac{d}{ds}\right)^{1}{}^{\rho}\mathcal{I}_{0^{+}}^{1-(\alpha-1)}u\left(s\right)\,=\,{}^{\rho}\mathcal{D}_{0^{+}}^{\alpha-1}u\left(s\right). \tag{3.3}$$

Then

$$\begin{aligned}{}^{\rho}\mathcal{I}_{0^{+}}^{\alpha}\ {}^{\rho}\mathcal{D}_{0^{+}}^{\alpha}u\left(t\right) &= \underbrace{t^{1-\rho}\frac{d}{dt}\left(\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\frac{d}{ds}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)ds\right)}_{\psi}\\ &\quad-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{1-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)},\end{aligned}$$

where

$$\begin{aligned}\psi &= t^{1-\rho}\frac{d}{dt}\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\left(\left[\left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)\right]_{0}^{t}\right.\\ &\quad\left.+\rho\left(\alpha-1\right)\int_{0}^{t}s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-2}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)ds\right)\\ &= t^{1-\rho}\frac{d}{dt}\left(\frac{\rho^{2-\alpha}}{\Gamma\left(\alpha-1\right)}\int_{0}^{t}s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-2}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(s\right)ds\right.\\ &\quad\left.-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= t^{1-\rho}\frac{d}{dt}\left({}^{\rho}\mathcal{I}_{0^{+}}^{\alpha-1}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(t\right)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= t^{1-\rho}\frac{d}{dt}\left({}^{\rho}\mathcal{I}_{0^{+}}^{1}u\left(t\right)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= u\left(t\right)-\frac{\rho^{2-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha-1\right)}t^{\rho(\alpha-2)}.\end{aligned}$$

Finally, for $1<\alpha\leq 2$, we have:

$${}^{\rho}\mathcal{I}_{0^{+}}^{\alpha}\ {}^{\rho}\mathcal{D}_{0^{+}}^{\alpha}u\left(t\right)=u\left(t\right)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{1-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}-\frac{\rho^{2-\alpha}\,{}^{\rho}\mathcal{I}_{0^{+}}^{2-\alpha}u\left(0^{+}\right)}{\Gamma\left(\alpha-1\right)}t^{\rho(\alpha-2)}. \tag{3.4}$$

As

$${}^{\rho}\mathcal{I}_{0^{+}}^{\alpha}t^{\mu}=\frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)}t^{\mu+\alpha\rho},\ \forall\mu>-\rho,$$

we use (3.2) , (3.3) , to prove that

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right] = C_1 \frac{\rho^{-(1-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(1-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)} t^{\rho(\alpha-1)+(1-\alpha)\rho} = C_1\rho^{\alpha-1}\Gamma(\alpha), \tag{3.5} $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right] = C_2 \, {}^{\rho}\mathcal{D}_{0^+}^{\alpha-1} t^{\rho(\alpha-2)} = C_2 \, {}^{\rho}\mathcal{D}_{0^+}^{\alpha-1} t^{\rho((\alpha-1)-1)} = 0, \tag{3.6} $$

for some $C_1, C_2 \in \mathbb{R}$, and

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right] = C_1 \frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)} t^{\rho(\alpha-1)+(2-\alpha)\rho} = C_1\rho^{\alpha-2}\Gamma(\alpha)\, t^{\rho} \tag{3.7} $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right] = C_2 \frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-2)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-2)}{\rho}\right)} t^{\rho(\alpha-2)+(2-\alpha)\rho} = C_2\rho^{\alpha-2}\Gamma(\alpha-1). \tag{3.8} $$

Then, for $u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}$, we have respectively:

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u\left(0^+\right) = {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right]\left(0^+\right) + {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right]\left(0^+\right) = C_1\rho^{\alpha-1}\Gamma(\alpha), \tag{3.9} $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha} u\left(0^+\right) = {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right]\left(0^+\right) + {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right]\left(0^+\right) = C_2\rho^{\alpha-2}\Gamma(\alpha-1). \tag{3.10} $$

From (3.4) , (3.5) , (3.6) , (3.7) , (3.8) , (3.9) and (3.10) we get (3.1) . □

In the following lemma, we define the integral solution of the boundary value problem (1.1)-(1.2) .

Lemma 3.2. *Let* $\alpha, \rho \in \mathbb{R}^+$, *be such that* $1 < \alpha \leq 2$. *We give* ${}^{\rho}\mathcal{D}_{0^+}^{\alpha} u \in C[0,T]$, *and* $f(t,u)$ *is a continuous function. Then the boundary value problem* (1.1)-(1.2) , *is equivalent to the fractional integral equation*

$$ u(t) = \beta \int_0^T G(t,s)\, f(s, u(s))\, ds, \ t \in [0,T], $$

where

$$ G(t,s) = \begin{cases} \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1} - \left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\right], & 0 \leq s \leq t \leq T, \\ \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1}, & 0 \leq t \leq s \leq T, \end{cases} \tag{3.11} $$

is the Green's function associated with the boundary value problem (1.1)-(1.2) .

Proof. Let $\alpha, \rho \in \mathbb{R}^+$, be such that $1 < \alpha \leq 2$. We apply Lemma 3.1 to reduce the fractional equation (1.1) to an equivalent fractional integral equation. It is easy to prove the operator ${}^{\rho}\mathcal{I}^{\alpha}_{0+}$ has the linearity property for all $\alpha > 0$ after direct integration. Then by applying ${}^{\rho}\mathcal{I}^{\alpha}_{0+}$ to equation (1.1), we get

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u\left(t\right) + \beta\ {}^{\rho}\mathcal{I}^{\alpha}_{0+}f\left(t,u\left(t\right)\right) = 0. $$

From Lemma 3.1, we find for $1 < \alpha \leq 2$,

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u\left(t\right) = u\left(t\right) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}, $$

for some $C_1, C_2 \in \mathbb{R}$. Then, the integral solution of the equation (1.1) is:

$$ u\left(t\right) = -\frac{\beta\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \frac{s^{\rho-1}f\left(s,u\left(s\right)\right)}{\left(t^{\rho}-s^{\rho}\right)^{1-\alpha}}ds - C_1 t^{\rho(\alpha-1)} - C_2 t^{\rho(\alpha-2)}. \tag{3.12} $$

The conditions (1.2) imply that:

$$ \begin{cases} u\left(0\right) = 0 = 0 - 0 - \lim\limits_{t\to 0} C_2 t^{\rho(\alpha-2)} & \Rightarrow \quad C_2 = 0, \\ u\left(T\right) = 0 = -\frac{\beta\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^T \frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds - C_1 T^{\rho(\alpha-1)} & \Rightarrow \quad C_1 = -\frac{\beta\rho^{1-\alpha}}{T^{\rho(\alpha-1)}\Gamma(\alpha)}\int\limits_0^T \frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds. \end{cases} $$

The integral equation (3.12) is equivalent to:

$$ u\left(t\right) = -\frac{\beta\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \frac{s^{\rho-1}f\left(s,u\left(s\right)\right)}{\left(t^{\rho}-s^{\rho}\right)^{1-\alpha}}ds + \frac{\beta t^{\rho(\alpha-1)}\rho^{1-\alpha}}{T^{\rho(\alpha-1)}\Gamma\left(\alpha\right)}\int_0^T \frac{s^{\rho-1}f\left(s,u\left(s\right)\right)}{\left(T^{\rho}-s^{\rho}\right)^{1-\alpha}}ds. $$

Therefore, the unique solution of problem (1.1)-(1.2) is:

$$ \begin{aligned} u\left(t\right) &= \beta\int_0^t \frac{\rho^{1-\alpha}s^{\rho-1}\left[\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1} - \left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\right]}{\Gamma\left(\alpha\right)}f\left(s,u\left(s\right)\right)ds \\ &\quad +\beta\int_t^T \frac{\rho^{1-\alpha}s^{\rho-1}\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(s,u\left(s\right)\right)ds \\ &= \beta\int_0^T G\left(t,s\right)f\left(s,u\left(s\right)\right)ds. \end{aligned} $$

The proof is complete. □

3.1. Application of Guo-Krasnosel'skii fixed point theorem

In this part, we assume that $\beta > 0$ and $0 < \rho \leq 1$. We impose some conditions on f, which allow us to obtain some results on existence of positive solutions for the boundary value problem (1.1)-(1.2).

We note that $u\left(t\right)$ is a solution of (1.1)-(1.2) if and only if:

$$ u\left(t\right) = \beta\int_0^T G\left(t,s\right)f\left(s,u\left(s\right)\right)ds,\ \ t\in\left[0,T\right]. $$

Now we prove some properties of the Green's function $G\left(t,s\right)$ given by (3.11).

Lemma 3.3. *Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, then the Green's function $G(t,s)$ given by* (3.11) *satisfies:*

(1) $G(t,s) > 0$ *for* $t, s \in (0,T)$.

(2) $\max\limits_{0\leq t\leq T} G(t,s) = G(s,s)$, *for each* $s \in [0,T]$.

(3) *For any* $t \in [0,T]$,

$$G(t,s) \geq b(t)\, G(s,s), \text{ for any } \frac{T}{8} \leq s \leq T \text{ and some } b \in C[0,T]. \tag{3.13}$$

Proof. (1) Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$. In the case $0 < t \leq s < T$, we have:

$$\frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} > 0.$$

Moreover, for $0 < s \leq t < T$, we have $\frac{t^\rho}{T^\rho} < 1$, then $\frac{t^\rho}{T^\rho} s^\rho < s^\rho$ and $t^\rho - \frac{t^\rho}{T^\rho} s^\rho > t^\rho - s^\rho$, thus

$$t^\rho - \frac{t^\rho}{T^\rho} s^\rho = \frac{t^\rho}{T^\rho}(T^\rho - s^\rho) > t^\rho - s^\rho \Rightarrow \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1} > 0,$$

which imply that $G(t,s) > 0$ for any $t, s \in (0,T)$.
(2) To prove that

$$\max_{0\leq t\leq T} G(t,s) = G(s,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}, \ \forall s \in [0,T], \tag{3.14}$$

we choose

$$g_1(t,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1}\right],$$

$$g_2(t,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}.$$

Indeed, we put $\max\limits_{0\leq t\leq T} G(t,s) = G(t^*,s)$, where $0 \leq t^* \leq T$. Then, we get for some $0 < t_1 < t_2 < T$, that

$$\begin{aligned}
\max_{0\leq t\leq T} G(t,s) &= \begin{cases} g_1(t^*,s), & s \in [0,t_1], \\ \max\{g_1(t^*,s), g_2(t^*,s)\}, & s \in [t_1,t_2], \\ g_2(t^*,s), & s \in [t_2,T], \end{cases} \\
&= \begin{cases} g_1(t^*,s), & s \in [0,r], \\ g_2(t^*,s), & s \in [r,T], \end{cases}
\end{aligned}$$

where $r \in [t_1,t_2]$, is the unique solution of equation

$$g_1(t^*,s) = g_2(t^*,s) \Leftrightarrow t^* = s,$$

which shows the equality (3.14).
(3) In the following, we divide the proof into two-part, to show the existence $b \in C[0,T]$, such that

$$G(t,s) \geq b(t)\, G(s,s)\,,\ \text{for any } \frac{T}{8} \leq s \leq T.$$

(i) Firstly, if $0 \leq t \leq s \leq T$, we see that $\frac{G(t,s)}{G(s,s)}$ is decreasing with respect to s. Consequently

$$\frac{G(t,s)}{G(s,s)} = \frac{\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}}{\left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}} = \left(\frac{t}{s}\right)^{\rho(\alpha-1)} \geq \left(\frac{t}{T}\right)^{\rho(\alpha-1)} = b_1(t)\,,\ \forall t \in [0,s]\,.$$

(ii) In the same way, if $0 \leq s \leq t \leq T$, we have $\frac{s^\rho}{T^\rho} < \frac{t^\rho}{T^\rho} \leq 1$, $\left(\frac{t^\rho}{T^\rho}\right)^{\alpha-2} \geq 1$, $\forall \alpha \in (1,2]$, and

$$\begin{aligned}
G(t,s) &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1}\right] \\
&= \frac{(\alpha-1)\,\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \int_{t^\rho - s^\rho}^{\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)} \tau^{\alpha-2} d\tau \\
&\geq \frac{(\alpha-1)\,\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left(\frac{t^\rho}{T^\rho}\right)^{\alpha-2} (T^\rho - s^\rho)^{\alpha-2} \left(\frac{t^\rho}{T^\rho}(T^\rho - s^\rho) - (t^\rho - s^\rho)\right) \\
&\geq \frac{(\alpha-1)\,\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} (T^\rho - s^\rho)^{\alpha-1} \frac{s^\rho (T^\rho - t^\rho)}{T^\rho (T^\rho - s^\rho)}.
\end{aligned}$$

As $0 < \rho \leq 1$, we get

$$T^\rho - t^\rho = \rho \int_t^T \tau^{\rho-1} d\tau \geq \rho T^{\rho-1}(T-t)\,,\ \text{and } T^\rho - s^\rho = \rho \int_s^T \tau^{\rho-1} d\tau \leq \rho s^{\rho-1}(T-s)\,.$$

Therefore

$$\begin{aligned}
\frac{G(t,s)}{G(s,s)} &\geq \frac{\frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} (T^\rho - s^\rho)^{\alpha-1} \frac{s^\rho(T^\rho - t^\rho)}{T^\rho(T^\rho - s^\rho)}}{\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}} = (\alpha-1)\frac{s^\rho (T^\rho - t^\rho)}{T^\rho (T^\rho - s^\rho)} \left(\frac{T^\rho}{s^\rho}\right)^{\alpha-1} \\
&\geq (\alpha-1)\frac{s(T-t)}{T(T-s)} \\
&\geq (\alpha-1)\frac{s(T-t)}{T^2}.
\end{aligned}$$

Finally, for $s \in \left[\frac{T}{8}, t\right]$, we have:

$$\frac{G(t,s)}{G(s,s)} \geq \frac{(\alpha-1)(T-t)}{8T} = b_2(t)\,.$$

It is clear that $b_1(t)$ and $b_2(t)$ are positive functions, it is enough to choose:

$$b(t) = \begin{cases} \left(\frac{t}{T}\right)^{\rho(\alpha-1)}, & \text{for } t \in [0, \bar{t}], \\ \frac{(\alpha-1)(T-t)}{8T}, & \text{for } t \in [\bar{t}, T], \end{cases} \tag{3.15}$$

where $\bar{t} \in (0, T)$ is the unique solution of the equation $b_1(t) = b_2(t)$. We see that

$$b(t) \le \bar{b} = b(\bar{t}) = \left(\frac{\bar{t}}{T}\right)^{\rho(\alpha-1)} = \frac{(\alpha-1)(T-\bar{t})}{8T} < 1 \text{ for all } t \in [0, T].$$

Finally, we have $\forall s \in \left[\frac{T}{8}, T\right]$,

$$G(t, s) \ge b(t)\, G(s, s), \ \forall t \in [0, T].$$

The proof is complete. □

Lemma 3.4. *Let* $1 < \alpha \le 2$ *and* $0 < \rho \le 1$, *then there exists a positive constant*

$$\lambda = 1 + \frac{8^{\rho\alpha} L(\alpha+1)\left[8^{\rho\alpha} - (8^\rho - 1)^\alpha\right]}{h(8^\rho - 1)^\alpha \left[8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho-1)\right]}, \text{ for some } h, L > 0,$$

such that

$$\int_0^T G(s, s) f(s, u(s))\, ds \le \lambda \int_{\frac{T}{8}}^T G(s, s) f(s, u(s))\, ds. \tag{3.16}$$

Proof. As $f(t, u(t)) \ge h$, for any $t \in [0, T]$, we get

$$\begin{aligned} \int_{\frac{T}{8}}^T G(s,s) f(s, u(s))\, ds &\ge h \int_{\frac{T}{8}}^T \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} ds \\ &\ge -\frac{h}{\alpha\rho^\alpha T^{\rho(\alpha-1)}\Gamma(\alpha)} \int_{\frac{T}{8}}^T s^{\rho(\alpha-1)}\left[-\rho\alpha s^{\rho-1}(T^\rho - s^\rho)^{\alpha-1}\right] ds. \end{aligned}$$

The integral by part gives:

$$\begin{aligned} \int_{\frac{T}{8}}^T G(s,s) f(s, u(s))\, ds &\ge \frac{h\left[\frac{T^{\rho(\alpha-1)}}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha + \rho(\alpha-1)\int_{\frac{T}{8}}^T s^{\rho(\alpha-1)-1}(T^\rho - s^\rho)^\alpha ds\right]}{\rho^\alpha T^{\rho(\alpha-1)}\Gamma(\alpha+1)} \\ &\ge \frac{h\left[\frac{T^\rho}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha + \rho(\alpha-1)\int_{\frac{T}{8}}^T \frac{s^{\rho(\alpha-2)}}{T^{\rho(\alpha-2)}} s^{\rho-1}(T^\rho - s^\rho)^\alpha ds\right]}{\rho^\alpha T^\rho \Gamma(\alpha+1)} \\ &\ge \frac{h\left[\frac{T^\rho}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha - \frac{\alpha-1}{\alpha+1}\int_{\frac{T}{8}}^T \left[-\rho(\alpha+1)s^{\rho-1}(T^\rho - s^\rho)^\alpha\right] ds\right]}{\rho^\alpha T^\rho \Gamma(\alpha+1)} \\ &\ge \frac{h T^{\rho\alpha}(8^\rho - 1)^\alpha}{\rho^\alpha 8^{\rho\alpha}\Gamma(\alpha+1)}\left[\frac{8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho - 1)}{8^{\rho\alpha}(\alpha+1)}\right]. \end{aligned}$$

Then

$$\frac{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}{hT^{\rho\alpha}(8^{\rho}-1)^{\alpha}}\left[\frac{8^{\rho\alpha}(\alpha+1)}{8^{\rho}(\alpha+1)+8^{\rho(\alpha-1)}(\alpha-1)(8^{\rho}-1)}\right]\int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds\geq 1. \tag{3.17}$$

On the other hand, if $\max\limits_{0\leq t\leq T} f(t,u)$ is bounded for $u\in[0,\infty)$, then there exists $L_0>0$, such that

$$|f(t,u(t))|\leq L_0,\ \forall t\in[0,T].$$

In the similar way, if $\max\limits_{0\leq t\leq T} f(t,u)$ is unbounded for $u\in[0,\infty)$, then there exists $M_0>0$, such that

$$\sup_{0\leq u\leq M_0}\max_{0\leq t\leq T}|f(t,u(t))|\leq L_1,\ \text{for some } L_1>0.$$

In all cases, for $L=\max\{L_0,L_1\}$, we have:

$$\int_0^{\frac{T}{8}}G(s,s)f(s,u(s))\,ds\leq L\int_0^{\frac{T}{8}}G(s,s)\,ds\leq\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{8^{\rho\alpha}\rho^{\alpha}\Gamma(\alpha+1)}.$$

From (3.17), we get

$$\begin{aligned}\int_0^{T}G(s,s)f(s,u(s))\,ds &= \int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds+\int_0^{\frac{T}{8}}G(s,s)f(s,u(s))\,ds\\ &\leq \int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds+\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}\\ &\leq \int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds\\ &\quad+\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}\times\frac{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}{hT^{\rho\alpha}(8^{\rho}-1)^{\alpha}}\\ &\quad\times\left[\frac{8^{\rho\alpha}(\alpha+1)}{8^{\rho}(\alpha+1)+8^{\rho(\alpha-1)}(\alpha-1)(8^{\rho}-1)}\right]\\ &\quad\times\int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds\\ &\leq \lambda\int_{\frac{T}{8}}^{T}G(s,s)f(s,u(s))\,ds.\end{aligned}$$

□

Let us define the cone P by:

$$P=\left\{u\in C[0,T]\ \mid u(t)\geq\frac{b(t)}{\lambda}\|u\|,\ \forall t\in[0,T]\right\}. \tag{3.18}$$

Lemma 3.5. *Let $\mathcal{A}: P \to C[0,T]$ be an integral operator defined by:*

$$\mathcal{A}u(t) = \beta \int_0^T G(t,s) f(s,u(s))\, ds, \tag{3.19}$$

equipped with standard norm

$$\|\mathcal{A}u\| = \max_{0 \le t \le T} |\mathcal{A}u(t)|.$$

Then $\mathcal{A}(P) \subset P$.

Proof. For any $u \in P$, we have from (3.13), (3.16) and (3.18), that

$$\begin{aligned}
\mathcal{A}u(t) &= \beta \int_0^T G(t,s) f(s,u(s))\, ds \ge \beta b(t) \int_{\frac{T}{8}}^T G(s,s) f(s,u(s))\, ds \\
&\ge \frac{\beta b(t)}{\lambda} \int_0^T G(s,s) f(s,u(s))\, ds \\
&\ge \frac{b(t)}{\lambda} \max_{0 \le t \le T} \left(\beta \int_0^T G(t,s) f(s,u(s))\, ds \right) \\
&\ge \frac{b(t)}{\lambda} \|\mathcal{A}u\|, \ \forall t \in [0,T].
\end{aligned}$$

Thus $\mathcal{A}(P) \subset P$. The proof is complete. □

Lemma 3.6. *$\mathcal{A}: P \to P$ is a completely continuous operator.*

Proof. In view of continuity of $G(t,s)$ and $f(t,u)$, the operator $\mathcal{A}: P \to P$ is a continuous.
Let $\Omega \subset P$ be a bounded. Then there exists a positive constant $M > 0$, such that:

$$\|u\| \le M, \ \forall u \in \Omega.$$

By choice

$$L = \sup_{0 \le u \le M} \max_{0 \le t \le T} |f(t,u)| + 1.$$

In this case, we get $\forall u \in \Omega$,

$$\begin{aligned}
|\mathcal{A}u(t)| &= \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \le \beta \int_0^T |G(t,s) f(s,u(s))|\, ds \\
&\le \beta L \int_0^T G(s,s)\, ds \le \frac{\beta L}{\rho^{\alpha-1}\Gamma(\alpha)} \int_0^T s^{\rho-1} (T^\rho - s^\rho)^{\alpha-1}\, ds \\
&\le \frac{\beta L T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)}.
\end{aligned}$$

Consequently, $|\mathcal{A}u(t)| \leq \frac{\beta L T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}$, $\forall u \in \Omega$. Hence, $\mathcal{A}(\Omega)$ is bounded.
Now, for $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, we give:

$$\delta(\varepsilon) = \left(\frac{\rho^{\alpha}\Gamma(\alpha)}{T^{\rho}\beta L}\varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}, \text{ for some } \varepsilon > 0.$$

Then $\forall u \in \Omega$, and $t_1, t_2 \in [0, T]$, where $t_1 < t_2$, and $t_2 - t_1 < \delta$, we find $|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| < \varepsilon$.
Consequently, for $0 \leq s \leq t_1 < t_2 \leq T$, we have:

$$\begin{aligned} G(t_2,s) - G(t_1,s) &= \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left(\frac{T^{\rho}-s^{\rho}}{T^{\rho}}\right)^{\alpha-1}\right. \\ &\quad \left. - \left[(t_2^{\rho}-s^{\rho})^{\alpha-1} - (t_1^{\rho}-s^{\rho})^{\alpha-1}\right]\right] \\ &< \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left(\frac{T^{\rho}-s^{\rho}}{T^{\rho}}\right)^{\alpha-1} \\ &< \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]. \end{aligned}$$

In the same way, for $0 \leq t_1 \leq s < t_2 \leq T$ or $0 \leq t_1 < t_2 \leq s \leq T$, we have:

$$G(t_2,s) - G(t_1,s) < \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right].$$

Then

$$\begin{aligned} |\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &= \left|\beta\int_0^T [G(t_2,s) - G(t_1,s)]\, f(s,u(s))\, ds\right| \\ &\leq \beta L\int_0^T |G(t_2,s) - G(t_1,s)|\, ds \\ &< \beta L\int_0^T \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] ds \\ &< \frac{\beta L\rho^{1-\alpha}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left[\frac{1}{\rho}s^{\rho}\right]_0^T. \end{aligned}$$

Finally

$$|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]. \tag{3.20}$$

In the following, we divide the proof into three cases.
(a) If $\delta \leq t_1 < t_2 \leq T$, we have:

$$\delta \leq t_1 < t_2 \Leftrightarrow t_2^{\rho(\alpha-2)} < t_1^{\rho(\alpha-2)} \leq \delta^{\rho(\alpha-2)}, \text{ and } t_2^{\rho-1} < t_1^{\rho-1} \leq \delta^{\rho-1}.$$

Thus

$$t_2^{\rho} - t_1^{\rho} = t_2 t_2^{\rho-1} - t_1 t_1^{\rho-1} < t_2 t_2^{\rho-1} - t_1 t_2^{\rho-1} = t_2^{\rho-1}(t_2 - t_1) < \delta^{\rho-1}(t_2 - t_1) < \delta^{\rho}.$$

In similar way

$$\begin{aligned} t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} &= t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_1^{\rho(\alpha-2)} < t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_2^{\rho(\alpha-2)} = t_2^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) \\ &< \delta^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) \\ &< \delta^{\rho(\alpha-1)}. \end{aligned}$$

Then, the inequality (3.20) gives:

$$\begin{aligned} \left|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)\right| &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \delta^{\rho(\alpha-1)} \\ &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[\left(\frac{\rho^{\alpha}\Gamma(\alpha)}{T^{\rho}\beta L}\varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}\right]^{\rho(\alpha-1)} \\ &< \varepsilon. \end{aligned} \tag{3.21}$$

(b) If $t_1 \leq \delta < t_2 < 2\delta$, we have:

$$t_1 \leq \delta < t_2 \Leftrightarrow t_2^{\rho(\alpha-2)} < \delta^{\rho(\alpha-2)} \leq t_1^{\rho(\alpha-2)},$$

and

$$\begin{aligned} t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} &= t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_1^{\rho(\alpha-2)} < t_2^{\rho} \delta^{\rho(\alpha-2)} - t_1^{\rho} \delta^{\rho(\alpha-2)} \\ &< \delta^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) < \delta^{\rho(\alpha-1)}. \end{aligned}$$

Also, we find the same result (3.21).
(c) If $t_1 < t_2 \leq \delta$, we have:

$$\begin{aligned} \left|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)\right| &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} t_2^{\rho(\alpha-1)} \\ &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \delta^{\rho(\alpha-1)} \\ &< \varepsilon. \end{aligned}$$

By the means of the Ascoli-Arzel Theorem 2.11, we have $\mathcal{A} : P \to P$ is completely continuous. □

We define some important constants

$$\begin{aligned} &F_0 = \lim_{u \to 0^+} \max_{t \in [0,T]} \tfrac{f(t,u)}{u}, \quad F_\infty = \lim_{u \to +\infty} \max_{t \in [0,T]} \tfrac{f(t,u)}{u}, \\ &f_0 = \lim_{u \to 0^+} \min_{t \in [0,T]} \tfrac{f(t,u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \min_{t \in [0,T]} \tfrac{f(t,u)}{u}, \\ &\omega_1 = \int_0^T G(s,s)\, ds, \qquad \omega_2 = \tfrac{\bar{b}}{\lambda^2} \int_0^T G(s,s)\, b(s)\, ds. \end{aligned}$$

Assume that $\frac{1}{\omega_2 f_\infty} = 0$ if $f_\infty \to \infty$, $\frac{1}{\omega_1 F_0} = \infty$ if $F_0 \to 0$, $\frac{1}{\omega_2 f_0} = 0$ if $f_0 \to \infty$, and $\frac{1}{\omega_1 F_\infty} = \infty$ if $F_\infty \to 0$.

Theorem 3.7. *If $\omega_2 f_\infty > \omega_1 F_0$ holds, then for each:*

$$\beta \in \left(\left(\omega_2 f_\infty\right)^{-1}, \left(\omega_1 F_0\right)^{-1} \right), \tag{3.22}$$

the boundary value problem (1.1)-(1.2) *has at least one positive solution.*

Proof. Let β satisfies (3.22) and $\varepsilon > 0$, be such that

$$\left(\left(f_\infty - \varepsilon\right)\omega_2\right)^{-1} \leq \beta \leq \left(\left(F_0 + \varepsilon\right)\omega_1\right)^{-1}. \tag{3.23}$$

From the definition of F_0, we see that there exists $r_1 > 0$, such that

$$f\left(t, u\right) \leq \left(F_0 + \varepsilon\right) u, \ \forall t \in [0, T], \ 0 < u \leq r_1. \tag{3.24}$$

Consequently, for $u \in P$ with $\|u\| = r_1$, we have from $(3.23), (3.24)$, that

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G\left(t, s\right) f\left(s, u\left(s\right)\right) ds \right| \\ &\leq \beta \int_0^T G\left(s, s\right) \left(F_0 + \varepsilon\right) u\left(s\right) ds \\ &\leq \beta \left(F_0 + \varepsilon\right) \|u\| \int_0^T G\left(s, s\right) ds \\ &\leq \beta \left(F_0 + \varepsilon\right) \|u\| \omega_1 \\ &\leq \|u\|. \end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in C[0, T] : \ \|u\| < r_1\}$, then

$$\|\mathcal{A}u\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_1. \tag{3.25}$$

By definition of f_∞, there exists $r_3 > 0$, such that

$$f\left(t, u\right) \geq \left(f_\infty - \varepsilon\right) u, \ \forall t \in [0, T], \ u \geq r_3. \tag{3.26}$$

Therefore, for $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, we have from $(3.23), (3.26)$, that

$$\begin{aligned} \|\mathcal{A}u\| &\geq \mathcal{A}u\left(\bar{t}\right) = \beta \int_0^T G\left(\bar{t}, s\right) f\left(s, u\left(s\right)\right) ds \geq \beta \int_{\frac{T}{8}}^T b\left(\bar{t}\right) G\left(s, s\right) f\left(s, u\left(s\right)\right) ds \\ &\geq \frac{\beta \bar{b}}{\lambda} \int_0^T G\left(s, s\right) f\left(s, u\left(s\right)\right) ds \geq \frac{\beta \bar{b}}{\lambda} \int_0^T G\left(s, s\right) \left[\left(f_\infty - \varepsilon\right) u\left(s\right)\right] ds, \quad \forall t \in [0, T]. \end{aligned}$$

By definition of P in (3.18), we have:

$$\begin{aligned} \|\mathcal{A}u\| &\geq \frac{\beta \bar{b}\left(f_\infty - \varepsilon\right)}{\lambda^2} \|u\| \int_0^T G\left(s, s\right) b\left(s\right) ds \\ &\geq \beta\left(f_\infty - \varepsilon\right) \|u\| \omega_2 \\ &\geq \|u\|. \end{aligned}$$

If we set $\Omega_2 = \{u \in C[0,T] : \|u\| < r_2\}$, then

$$\|\mathcal{A}u\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{3.27}$$

Now, from (3.25), (3.27), and Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that u is a positive solution of (1.1)-(1.2). The proof is complete. □

Theorem 3.8. *If $\omega_2 f_0 > \omega_1 F_\infty$ holds, then for each:*

$$\beta \in \left((\omega_2 f_0)^{-1}, (\omega_1 F_\infty)^{-1} \right), \tag{3.28}$$

the boundary value problem (1.1)-(1.2) *has at least one positive solution.*

Proof. Let β satisfies (3.28) and $\varepsilon > 0$, be such that

$$((f_0 - \varepsilon)\omega_2)^{-1} \leq \beta \leq ((F_\infty + \varepsilon)\omega_1)^{-1}. \tag{3.29}$$

From definition of f_0, we see that there exists $r_1 > 0$, such that

$$f(t,u) \geq (f_0 - \varepsilon) u, \ \forall t \in [0,T], \ 0 < u \leq r_1.$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the proof's second part of Theorem 3.7, we can get that $\|\mathcal{A}u\| \geq \|u\|$. Then, if we choose $\Omega_1 = \{u \in C[0,T] : \ \|u\| < r_1\}$, thus

$$\|\mathcal{A}u\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_1. \tag{3.30}$$

Next, and by definition of F_∞, we may choose $R_1 > 0$, such that

$$f(t,u) \leq (F_\infty + \varepsilon) u, \text{ for } u \geq R_1. \tag{3.31}$$

We consider two cases:
1) If $\max\limits_{0 \leq t \leq T} f(t,u)$ is bounded for $u \in [0,\infty)$. Then, there exists some $L > 0$, such that

$$f(t,u) \leq L, \text{ for all } t \in [0,T], \ u \in P.$$

Let us denote by $r_3 = \max\{2r_1, \beta L \omega_1\}$, if $u \in P$ with $\|u\| = r_3$, then

$$\|\mathcal{A}u\| = \max_{0 \leq t \leq T} \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \leq \beta L \int_0^T G(s,s)\, ds = \beta L \omega_1 \leq r_3 = \|u\|.$$

Hence,

$$\|\mathcal{A}u\| \leq \|u\|, \text{ for } u \in \partial P_{r_3} = \{u \in P : \|u\| \leq r_3\}. \tag{3.32}$$

2) If $\max\limits_{0 \leq t \leq T} f(t,u)$ is unbounded for $u \in [0,\infty)$, then there exists some $r_4 = \max\{2r_1, R_1\}$, such that

$$f(t,u) \leq \max_{0 \leq t \leq T} f(t, r_4), \text{ for all } 0 < u \leq r_4, \ t \in [0,T].$$

Let $u \in P$ with $\|u\| = r_4$. Then, from (3.29), (3.31), we have:

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \leq \beta \int_0^T G(s,s)(F_\infty + \varepsilon) u(s)\, ds \\ &\leq \beta (F_\infty + \varepsilon) \|u\| \int_0^T G(s,s)\, ds = \beta (F_\infty + \varepsilon) \|u\| \omega_1 \\ &\leq \|u\|. \end{aligned}$$

Thus, (3.32) is also true for $u \in \partial P_{r_4}$.
In both cases 1 and 2, if we set $\Omega_2 = \{u \in C[0,T] : \ \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|\mathcal{A}u\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{3.33}$$

Now, from (3.30), (3.33), and Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that u is a positive solution of (1.1)-(1.2). The proof is complete. □

Theorem 3.9. *Suppose there exists* $r_2 > r_1 > 0$, *such that*

$$\sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(t,u) \leq \frac{r_2}{\beta \omega_1}, \text{ and } \inf_{0 \leq u \leq r_1} f(t,u) \geq \frac{r_1}{\beta \lambda \omega_2} b(t), \ \forall t \in [0,T]. \tag{3.34}$$

Then, the boundary value problem (1.1)-(1.2) *has a positive solution* $u \in P$, *with* $r_1 \leq \|u\| \leq r_2$.

Proof. Choose $\Omega_1 = \{u \in C[0,T] : \ \|u\| < r_1\}$. Then, for $u \in P \cap \partial\Omega_1$, we get

$$\begin{aligned} \|\mathcal{A}u\| &\geq \mathcal{A}u(\bar{t}) = \beta \int_0^T G(\bar{t},s) f(s,u(s))\, ds \geq \beta \int_{\frac{T}{8}}^T b(\bar{t}) G(s,s) f(s,u(s))\, ds \\ &\geq \frac{\beta \bar{b}}{\lambda} \int_0^T G(s,s) \inf_{0 \leq u \leq r_1} f(s,u(s))\, ds \geq \frac{\beta \bar{b}}{\lambda} \int_0^T G(s,s) \frac{r_1}{\beta \lambda \omega_2} b(s)\, ds \\ &\geq r_1 = \|u\|. \end{aligned}$$

On the other hand, choose $\Omega_2 = \{u \in C[0,T] : \ \|u\| < r_2\}$. Then, for $u \in P \cap \partial\Omega_2$, we get

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \leq \beta \int_0^T G(s,s) \sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(s,u(s))\, ds \\ &\leq \beta \int_0^T G(s,s) \frac{r_2}{\beta \omega_1} ds = r_2 = \|u\|. \end{aligned}$$

Now, from Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that u is a positive solution of (1.1)-(1.2). The proof is complete. □

3.2. Application of Banach fixed point theorem

In this part, we assume that $\beta \in \mathbb{R}$ and $\rho > 0$, and $f : [0,T] \times [0,\infty) \to [0,\infty)$ satisfies the conditions:

(H1) $f(t,u)$ is Lebesgue measurable function with respect to t on $[0,T]$,

(H2) $f(t,u)$ is continuous function with respect to u on $\mathbb{R}$.

Theorem 3.10. *Assume* (H1), (H2) *hold, and there exists a constant* $\sigma > 0$, *such that*

$$|f(t,u) - f(t,v)| \leq \sigma |u - v|, \text{ for almost every } t \in [0,T], \text{ and all } u,v \in C[0,T]. \tag{3.35}$$

If

$$|\beta| < \frac{\rho^{\alpha}\Gamma(\alpha+1)}{\sigma T^{\alpha\rho}}. \tag{3.36}$$

Then, there exists a unique solution of the boundary value problem (1.1)-(1.2) *on* $[0,T]$.

Proof. Assume that $|\beta| < \frac{\rho^{\alpha}\Gamma(\alpha+1)}{\sigma T^{\alpha\rho}}$, and consider the operator $\mathcal{A} : C[0,T] \to C[0,T]$ defined by (3.19) as follows

$$\mathcal{A}u(t) = \beta \int_0^T G(t,s) f(s,u(s))\, ds.$$

We shall show that $\mathcal{A}$ is a contraction mapping. In fact, for any $u,v \in C[0,T]$, we have

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{A}v(t)| &= \left| \beta \int_0^T G(t,s) [f(s,u(s)) - f(s,v(s))]\, ds \right| \\ &\leq |\beta| \int_0^T G(t,s) |f(s,u(s)) - f(s,v(s))|\, ds \\ &\leq |\beta| \sigma \int_0^T G(s,s) |u(s) - v(s)|\, ds, \end{aligned}$$

then

$$\begin{aligned} \|\mathcal{A}u - \mathcal{A}v\| &\leq |\beta| \sigma \|u - v\| \int_0^T G(s,s)\, ds \\ &\leq \frac{|\beta| \sigma T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} \|u - v\|. \end{aligned} \tag{3.37}$$

This imply from (3.37) that $\mathcal{A}$ is a contraction operator. As a consequence of Theorem 2.14, by Banach's contraction principle [5], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (1.1)-(1.2) on $[0,T]$. □

4. Examples

In this section, we present some examples to illustrate the usefulness of our main results.

Example 1. Consider the following boundary value problem

$$\begin{cases} {}^{1}\mathcal{D}_{0^+}^{\frac{3}{2}} u(t) + \beta (1+t) u(t) \ln(1+u(t)) = 0, & t \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{4.1}$$

Set $\beta > 0$ any finite positive real number, and

$$f(t,u) = (1+t) u \ln(1+u).$$

In this case, the function f is jointly continuous for any $t \in [0,1]$, and any $u > 0$.

We get

$$F_0 = \lim_{u \to 0^+} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0^+, \quad f_\infty = \lim_{u \to +\infty} \min_{t \in [0,T]} \frac{f(t,u)}{u} = \infty.$$

On the other hand, we get

$$\omega_1 = \int_0^1 G(s,s)\, ds = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \sqrt{s(1-s)} ds = \frac{1}{\frac{1}{2}\sqrt{\pi}} \frac{\pi}{8} = \frac{\sqrt{\pi}}{4}, \tag{4.2}$$

and

$$b(t) = \begin{cases} \sqrt{t} & \text{for } t \in [0, \bar{t}], \\ \frac{1-t}{16} & \text{for } t \in [\bar{t}, 1]. \end{cases} \tag{4.3}$$

Then

$$\omega_2 = \frac{\bar{b}}{\lambda^2 \Gamma\left(\frac{3}{2}\right)} \left[\int_0^{\bar{t}} s\sqrt{(1-s)} ds + \frac{1}{16} \int_{\bar{t}}^1 \sqrt{s}(1-s)^{\frac{3}{2}} ds \right] \simeq \frac{\bar{b}\sqrt{\pi}}{128\lambda^2}. \tag{4.4}$$

Where $\bar{t} \simeq 0,003876\ldots$ and $\bar{b} \simeq 0,062258\ldots$ and the choice of λ depends directly by choice of r_1, r_2 in (3.25), (3.27).

Because $\omega_1, \omega_2 > 0$, two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_\infty} = 0, \text{ and } \frac{1}{\omega_1 F_0} = \infty.$$

Then, the condition (3.22) is satisfied for any $0 < \beta < \infty$.

It follows from Theorem 3.7 that the problem (4.1) has at least one solution.

Example 2. Consider

$$\begin{cases} {}^{1}\mathcal{D}_{0^+}^{\frac{3}{2}} u(t) + \beta (1+t) u(t) \exp\left(\frac{1}{u(t)} - [u(t)]^2\right) = 0, & t \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{4.5}$$

Set $\beta > 0$ any finite positive real number, and

$$f(t,u) = (1+t)\, u \exp\left(\frac{1}{u} - u^2\right).$$

Clearly, for any $t \in [0,1]$ and any $u > 0$, the function f is jointly continuous.

Here, we have:

$$f_0 = \lim_{u\to 0^+} \min_{t\in[0,T]} \tfrac{f(t,u)}{u} = \infty, \quad F_\infty = \lim_{u\to +\infty} \max_{t\in[0,T]} \tfrac{f(t,u)}{u} = 0^+.$$

Also, we find the same function $b(t)$ in (4.3), and same constant ω_1, ω_2 respectively in (4.2), (4.4).

The choice of $\lambda > 1$ depends directly by choice of r_1, r_2 in (3.30), (3.33).

Because $\omega_1, \omega_2 > 0$, two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_0} = 0, \text{ and } \frac{1}{\omega_1 F_\infty} = \infty.$$

Then, the condition (3.28) is satisfied for any $0 < \beta < \infty$.

It follows from Theorem 3.8 that the problem (4.5) has at least one solution.

Example 3. Consider the following boundary value problem

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u(t) + \frac{(1+t)(1+u(t))}{\sqrt{\pi}} = 0, & t \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{4.6}$$

Set $\beta = \frac{1}{\sqrt{\pi}}$, and

$$f(t,u) = (1+t)(1+u).$$

The function f is jointly continuous for any $t \in [0,1]$ and any $u > 0$.

We find the same function $b(t)$ in (4.3), such that $0 \leq b(t) < 1$, and

$$\omega_1 = \int_0^1 G(s,s)\, ds = \frac{\sqrt{\pi}}{4}.$$

Choosing $r_1 = \frac{1}{10^4} < r_2 = 2$. Then, for all $t \in [0,1]$, we have:

$$h = 1 \leq f(t,u) \leq 6 = L.$$

In this case

$$\begin{aligned} \lambda &= 1 + \frac{8^{\rho\alpha} L(\alpha+1)\left[8^{\rho\alpha} - (8^\rho - 1)^\alpha\right]}{h(8^\rho - 1)^\alpha \left[8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho-1)\right]} \\ &= 1 + \frac{8^{\frac{3}{2}} \times 6 \times \frac{5}{2} \times \left(8^{\frac{3}{2}} - 7^{\frac{3}{2}}\right)}{7^{\frac{3}{2}} \times \left(8 \times \frac{5}{2} + \sqrt{8} \times \frac{7}{2}\right)} \\ &\simeq 3,517426\ldots \end{aligned}$$

Then

$$\omega_2 \simeq \frac{\bar{b}\sqrt{\pi}}{128\lambda^2} \simeq \frac{0,062258 \times \sqrt{\pi}}{128 \times 3,517426^2} \simeq \frac{3,9313\sqrt{\pi}}{10^5}.$$

It remains to show that the conditions in (3.34) , which is

$$\sup_{0\leq u\leq r_2} \max_{0\leq t\leq T} f(t,u) = 6 \leq \frac{r_2}{\beta\omega_1} \simeq 8,$$

and

$$\inf_{0\leq u\leq r_1} f_3(t,u) = 1 + t \geq \frac{r_1}{\beta\lambda\omega_2} b(t) \simeq 0,72317 \times b(t), \quad \forall t \in [0,1].$$

Are satisfied. It follows from Theorem 3.9 that the problem (4.6) has at least one solution.

Example 4. Let

$$\begin{cases} {}^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}} u(t) + \frac{\cos(t)[2+|u(t)|]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u(t)|]} = 0, \quad t \in \left[0, \frac{\pi}{4}\right], \\ u(0) = u\left(\frac{\pi}{4}\right) = 0. \end{cases} \tag{4.7}$$

Set $\beta = \frac{1}{\pi}$ and

$$f(t,u) = \frac{\cos(t)[2+|u|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u|]}, \quad t \in \left[0, \frac{\pi}{4}\right], \ u, v \in \mathbb{R}.$$

As $\sin(t)$, $\cos(t)$ are continuous positive functions $\forall t \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(t) \leq 1$, and $0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}$, then

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left| \frac{\cos(t)[2+|u|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u|]} - \frac{\cos(t)[2+|v|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|v|]} \right| \\ &= \left| \frac{\cos(t)}{\sqrt{2}\cos(t)+\sin(t)} \right| \left| \frac{2+|u|}{1+|u|} - \frac{2+|v|}{1+|v|} \right| \\ &\leq ||u| - |v|| \leq |u - v|. \end{aligned}$$

Hence, the condition (3.35) is satisfied with $\sigma = 1$. It remains to show that the condition (3.36)

$$0 < \beta = \frac{1}{\pi} \simeq 0,318309\ldots < \frac{\rho^\alpha \Gamma(\alpha+1)}{\sigma T^{\alpha\rho}} = \frac{\frac{2}{3}^{\frac{3}{2}} \times \Gamma\left(\frac{5}{2}\right)}{\frac{\pi}{4}} \simeq 0,921317\ldots$$

is satisfied. It follows from Theorem 3.10 that the problem (4.7) has a unique solution.

5. Conclusion

In this paper we have discussed the existence and the uniqueness of solutions for a class of nonlinear fractional differential equations with a boundary value, by using the properties of Guo-Krasnosel'skii and Banach fixed point theorems. The used differential operator is developed by Katugampola, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives into a single form.

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