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Contents

On a starlikeness of order α condition for meromorphic m-valent functions

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ABSTRACT: The aim of the paper is to provide sufficient conditions for starlikeness of order α for meromorphic m-valent functions in the punctured disc. The present work is based on some results involving differential subordinations.

1 Introduction and preliminaries

Let Σ_m denote the class of functions of the form

$$f(z) = \frac{1}{z^m} + \sum_{n=1}^{\infty} a_{m+n-1} z^{m+n-1}, \quad m \in \mathbb{N}^* \tag{1.1}$$

which are analytic and m-valent in the punctured disc

$$\dot{U} = \{z \in C : \ 0 < |z| < 1\} = U \setminus \{0\}.$$

A function $f \in \Sigma_m$ is said [1] to be in the class $\Omega_m(\alpha)$ of meromorphic m-valently starlike functions of order α in $\dot{U}$ if and only if

$$\text{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \dot{U}, \ 0 \leq \alpha < m, \ m \in \mathbb{N}^*. \tag{1.2}$$

We denote $\Omega_m(0) = \Omega_m^*$.

The following definitions and lemmas will be used in the next section.

Let $\mathcal{H}(U)$ denote the space of analytic functions in U. For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}_n = \{f \in \mathcal{H}(U) : \ f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots\} \tag{1.3}$$

and
$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U): \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \}. \tag{1.4}$$

For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U$$

if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(w(z)), \quad z \in U. \tag{1.5}$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Lemma 1.1 [2] *Let m be a positive integer and let α be real, with $0 \leq \alpha < m$. Let $q \in \mathcal{H}(U)$, with $q(0) = 0$, $q'(0) \neq 0$ and*

$$\mathrm{Re}\ \left(1 + \frac{zq''(z)}{q'(z)}\right) > \frac{\alpha}{m}. \tag{1.6}$$

Define the function h as

$$h(z) = mzq'(z) - \alpha q(z). \tag{1.7}$$

If $p \in \mathcal{H}_m$ and

$$zp'(z) - \alpha p(z) \prec h(z) \tag{1.8}$$

then $p(z) \prec q(z)$ and this result is sharp.

Lemma 1.2 [3] *Let $n \in \mathbb{N}^*$, let $\alpha \in [0, 1]$ and let*

$$M_n(\alpha) = \frac{n + 1 - \alpha}{\sqrt{(n + 1 - \alpha)^2 + \alpha^2} + 1 - \alpha}. \tag{1.9}$$

If the function $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k \tag{1.10}$$

satisfies the condition

$$|z^2 f'(z) + (1 - \alpha)zf(z) + \alpha| < M_n(\alpha), \quad z \in U \tag{1.11}$$

then

$$\mathrm{Re}\ \left\{-\frac{zf'(z)}{f(z)}\right\} > 0.$$

2 Main results

Theorem 2.1 *If $f \in \Sigma_m$, $m \in \mathbb{N}^*$, on the form*

$$f(z) = \frac{1}{z^m} + \sum_{k=m}^{\infty} a_k z^k$$

and satisfies the condition

$$|(1 - \alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M, \quad \alpha \in [0, 2) \tag{2.1}$$

then

$$|z^m f(z) - 1| < \frac{M}{m(2 - \alpha)} \tag{2.2}$$

and this result is sharp.

Proof. If we let

$$p(z) = z^m f(z) - 1 \tag{2.3}$$

then $p \in \mathcal{H}_{2m}$ and (2.1) can be rewritten as

$$|zp'(z) - \alpha m p(z)| < M \tag{2.4}$$

or

$$zp'(z) - \alpha m p(z) \prec Mz. \tag{2.5}$$

If we take in Lemma 1.1

$$q(z) = \frac{Mz}{m(2 - \alpha)}, \quad q \in \mathcal{H}(U),$$

with $q(0) = 0$, $q'(0) \neq 0$ and

$$\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \frac{\alpha}{2}$$

then from (1.7), $h(z) = Mz$ and the result follows from Lemma 1.1, that is $p(z) \prec q(z)$

$$z^m f(z) - 1 \prec \frac{Mz}{m(2 - \alpha)}$$

or

$$|z^m f(z) - 1| < \frac{M}{m(2 - \alpha)}.$$

$\square$

Theorem 2.2 *Let $m \in \mathbb{N}^*$, $0 \le \alpha < \dfrac{m}{m+1}$ and let*

$$M(m, \alpha) = \frac{m(2 - \alpha)(m - \alpha)}{m(1 - \alpha) - \alpha + m(m - \alpha)\sqrt{\alpha^2 + (2 - \alpha)^2}}. \tag{2.6}$$

If $f \in \Sigma_m$ satisfies the condition

$$|(1 - \alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M(m, \alpha) \tag{2.7}$$

then $f \in \Omega_m(\alpha)$.

Proof. Let

$$0 < M \le M(m, \alpha), \tag{2.8}$$

where $M(m, \alpha)$ is given by (2.6), and suppose that $f \in \Sigma_m$ satisfies the condition

$$|(1 - \alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M. \tag{2.9}$$

If we set

$$P(z) = z^m f(z), \tag{2.10}$$

then by Theorem 2.1 we obtain

$$|P(z) - 1| < \frac{M}{m(2 - \alpha)} \equiv R, \quad z \in U. \tag{2.11}$$

From (2.6), we easily deduce $R < 1$, which implies $P(z) \ne 0$, $z \in U$. Hence if we let

$$p(z) = -\alpha - \frac{zf'(z)}{f(z)}, \tag{2.12}$$

then $p(z) \in \mathcal{H}[m - \alpha, 2m]$ and (2.9) can be written in the form

$$|-p(z)P(z) + [m(1 - \alpha) - \alpha]P(z) + \alpha m| < M. \tag{2.13}$$

We claim that this inequality implies Re $p(z) > 0$, $z \in U$. If this is false, then there exists a point $z_0 \in U$, such that $p(z_0) = i\rho$, where ρ is real. We will show that at such a point the negation of condition (2.13) holds, that is

$$|-i\rho P(z_0) + [m(1 - \alpha) - \alpha]P(z_0) + \alpha m| \ge M, \tag{2.14}$$

for all real ρ.

If we let $P_0 = P(z_0)$, one obtains

$$|-i\rho P_0 + [m(1 - \alpha) - \alpha]P_0 + \alpha m|^2 = \rho^2 |P_0|^2 + [m(1 - \alpha) - \alpha]^2 |P_0|^2$$

$$+\alpha^2 m^2 + 2\alpha m[m(1 - \alpha) - \alpha]\text{Re } P_0 + 2\alpha m \rho \text{Im } P_0.$$

The inequality (2.14) is equivalent to

$$E \equiv \rho^2 |P_0|^2 + 2\alpha m \rho \text{Im } P_0 + [m(1 - \alpha) - \alpha]^2 |P_0|^2 + \alpha^2 m^2 + \tag{2.15}$$

$$+2\alpha m[m(1-\alpha)-\alpha]\operatorname{Re} P_0 - R^2 m^2(2-\alpha)^2 \geq 0.$$

Since from (2.11) we have

$$|P_0| > 1-R \quad \text{and} \quad \operatorname{Re} P_0 > 1-R,$$

from (2.11) and (2.15) one obtains

$$E \geq |P_0|^2\rho^2 + 2\alpha m\operatorname{Im} P_0\rho + [m(1-\alpha)-\alpha]^2(1-R)^2+$$

$$+\alpha^2 m^2 + 2\alpha m[m(1-\alpha)-\alpha](1-R) - R^2 m^2(2-\alpha)^2.$$

Hence $E \geq 0$ if

$$\alpha^2 m^2(\operatorname{Im} P_0)^2 \leq |P_0|^2\{[(m(1-\alpha)-\alpha)(1-R)+\alpha m]^2 - R^2 m^2(2-\alpha)^2\} \quad (2.16)$$

or

$$\alpha^2 m^2(\operatorname{Im} P_0)^2 \leq |P_0|^2\{[m-\alpha-[m(1-\alpha)-\alpha]R]^2 - R^2 m^2(2-\alpha)^2\}. \quad (2.17)$$

A simple geometric argument shows that the inequality (2.11) implies

$$(\operatorname{Im} P_0)^2 \leq R^2|P_0|^2 \quad (2.18)$$

By comparing (2.17) and (2.18) we deduce that (2.14) holds if

$$\alpha^2 m^2 R^2 \leq \{m-\alpha-[m(1-\alpha)-\alpha]R\}^2 - R^2 m^2(2-\alpha)^2 \quad (2.19)$$

or

$$R^2\{\alpha^2 m^2 + m^2(2-\alpha)^2 - [m(1-\alpha)-\alpha]^2\}+ \quad (2.20)$$

$$+2(m-\alpha)[m(1-\alpha)-\alpha]R - (m-\alpha)^2 \leq 0$$

This last inequality holds if $R \leq R_0$, where

$$R_0 = \frac{m-\alpha}{m(1-\alpha)-\alpha+m\sqrt{\alpha^2+(2-\alpha)^2}}, \quad 0 \leq \alpha < \frac{m}{m+1} \quad (2.21)$$

that is $M \leq M(m,\alpha)$.

Thus we have a contradiction of (2.13), therefore $\operatorname{Re} p(z) > 0$, $z \in U$ and $f \in \Omega_m(\alpha)$. $\square$

Remark 2.1 Note that for the special case $m=1$, $\alpha=0$, the value $M(1,0)=2/3$ is the same with that obtained from (1.9) Lemma 1.2: $M_1(0)=2/3$.

We obtain the following criterion of starlikeness for meromorphic m-valent functions.

Corollary 2.1 *Let $m \in \mathbb{N}^*$ and let $f \in \Sigma_m$ satisfies the condition*

$$|mz^m f(z) + z^{m+1}f'(z)| < \frac{2m}{m+1} \quad (2.22)$$

then $f \in \Omega_m^*$.

Since a function $f \in \Sigma_m$ can be written as

$$f(z) = \frac{1}{z^m} + g(z), \quad 0 < |z| < 1 \tag{2.23}$$

where $g \in \mathcal{H}_m$, Theorem 2.2 can be rewritten in the following equivalent form, that is useful for the other results.

Corollary 2.2 *Let $m \in \mathbb{N}^*$, $0 \le \alpha < \dfrac{m}{m+1}$ and let $f \in \Sigma_m$ have the form*

$$f(z) = \frac{1}{z^m} + g(z),$$

where $g \in \mathcal{H}_m$. If

$$|(1-\alpha)m z^m g(z) + z^{m+1} g'(z)| < M(m, \alpha), \quad z \in U \tag{2.24}$$

where $M(m, \alpha)$ is given by (2.6), then $f \in \Omega_m(\alpha)$.

This form has an interesting interpretation in terms of integral operators. If we let

$$h(z) = (1-\alpha)m z^m g(z) + z^{m+1} g'(z), \tag{2.25}$$

then

$$g(z) = \frac{1}{z^{(1-\alpha)m}} \int_0^z h(t) t^{-(1+\alpha m)} dt \tag{2.26}$$

which leads to the following result.

Corollary 2.3 *Let $h \in \mathcal{H}_{2m}$ and $M(m, \alpha)$ is given by (2.6) with $0 \le \alpha < \dfrac{m}{m+1}$. If h satisfies the condition*

$$|h(z)| \le M(m, \alpha), \quad z \in U \tag{2.27}$$

then

$$f(z) = \frac{1}{z} + \frac{1}{z^{(1-\alpha)m}} \int_0^z h(t) t^{-(1+\alpha m)} dt \in \Omega_m(\alpha). \tag{2.28}$$

Example 2.1 For the Corollary 2.3 we consider the following function

$$h(z) = a z^3 (z - \sin z) \tag{2.29}$$

Since $h \in \mathcal{H}_6$ we deduce that $m = 3$ and we choose for α a value such that $0 \le \alpha < \dfrac{m}{m+1}$. Let the value be $\alpha = \dfrac{2}{3}$. Then, from (2.28) we get

$$f(z) = \frac{1}{z} + \frac{a}{z} \int_0^z t^3 (t - \sin t) t^{-3} dt \tag{2.30}$$

or

$$f(z) = \frac{1}{z} \left(1 - 2a \sin^2 \frac{z}{2} \right) + \frac{az}{2}. \tag{2.31}$$

From (2.27) we obtain

$$|h(z)| \leq M(m, \alpha) = M\left(3, \frac{2}{3}\right).$$ (2.32)

The above inequality leads to the relation

$$|az^3||z - \sin z| \leq |a|\frac{e^2 + 2e - 1}{2e}.$$ (2.33)

The condition (2.32) will be satisfied if

$$|a|\frac{e^2 + 2e - 1}{2e} \leq \frac{21}{1 + 7\sqrt{20}},$$ (2.34)

and we obtain

$$|a| \leq \frac{42e}{(2e + e^2 - 1)(1 + 7\sqrt{20})} = 0.298\ldots$$

Hence, if we take $a = \dfrac{1}{4}$ we conclude that

$$f(z) = \frac{1}{z}\left(1 - \frac{1}{2}\sin^2\frac{z}{2}\right) + \frac{z}{8} \in \Omega_3\left(\frac{2}{3}\right).$$

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Generalized classes of uniformly convex functions

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ABSTRACT: In this paper we introduce some subclasses of analytic functions with varying argument of coefficients. These classes are defined in terms of the Hadamard product and generalize the well-known classes of uniformly convex functions. We investigate the coefficients estimates, distortion properties, radii of starlikeness and convexity for defined classes of functions

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Key Words and Phrases: *analytic functions, varying arguments, subordination, Hadamard product*

1 Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z \in \mathbb{C} : |z| < r\}.$$

and let $\mathcal{A}(p, k)$ $(p, k \in \mathbb{N} = \{1, 2, 3...\}, \ p < k)$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{U}; \ a_p > 0). \tag{1}$$

For multivalent fuction $f \in \mathcal{A}(p, k)$ the normalization

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \text{ and } \left.\frac{f(z)}{z^p}\right|_{z=0} = 1. \tag{2}$$

is clasical. One can obtain interesting results by applying normalization of the form

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \text{ and } \left.\frac{f(z)}{z^p}\right|_{z=\rho} = 1. \tag{3}$$

where ρ is a fixed point of the unit disk $\mathcal{U}$. In particular, for $p = 1$ we obtain Montel's normaliztion (*cf.* [1]). We see that for $\rho = 0$ the normalization (3) is the clasical.

We denote by $\mathcal{A}_\rho(p, k)$ the classes of functions $f \in \mathcal{A}(p, k)$ with the normalization (3). It will be called the class of functions with two fixed points.

Also, by $\mathcal{T}(p, k; \eta)$ ($\eta \in \mathbb{R}$) we denote the class of functions $f \in \mathcal{A}(p, k)$ of the form (1) for which

$$\arg(a_n) = \pi + (p - n)\eta \quad (n = k, k + 1, ...). \tag{4}$$

For $\eta = 0$ we obtain the class $\mathcal{T}(p, k; 0)$ of functions with negative coefficients. Moreover, we define

$$\mathcal{T}(p, k) := \bigcup_{\eta \in \mathbb{R}} \mathcal{T}(p, k; \eta). \tag{5}$$

The classes $\mathcal{T}(p, k)$ and $\mathcal{T}(p, k; \eta)$ are called the classes of functions with varying argument of coefficients. The class $\mathcal{T}(1, 2)$ was introduced by Silverman [2] (see also [3]).

Let $\alpha \in \langle 0, p \rangle$, $r \in (0, 1\rangle$. A function $f \in \mathcal{A}(p, k)$ is said to be *convex of order* α in $\mathcal{U}(r)$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \;\; (z \in \mathcal{U}(r)).$$

A function $f \in \mathcal{A}(p, k)$ is said to be *starlike of order α* in $\mathcal{U}(r)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \;\; (z \in \mathcal{U}(r)). \tag{6}$$

We denote by $\mathcal{S}^c(\alpha)$ the class of all functions $f \in \mathcal{A}(p, p+1)$, which are convex of order α in $\mathcal{U}$ and by $\mathcal{S}^*_p(\alpha)$ we denote the class of all functions $f \in \mathcal{A}(p, p+1)$, which are starlike of order α in $\mathcal{U}$. We also set

$$\mathcal{S}^c = \mathcal{S}^c_1(0) \text{ and } \mathcal{S}^* = \mathcal{S}^*_1(0).$$

It is easy to show that for a function f from the class $\mathcal{T}(p, k)$ the condition (6) is equivalent to the following

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha \quad (z \in \mathcal{U}(r)). \tag{7}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}(p, k)$. We define *the radius of starlikeness of order α* and *the radius of convexity of order α* for the class $\mathcal{B}$ by

$$R^*_\alpha(\mathcal{B}) = \inf_{f \in \mathcal{B}} \left(\sup\{r \in (0, 1] : f \text{ is starlike of order } \alpha \text{ in } \mathcal{U}(r)\}\right),$$

$$R^c_\alpha(\mathcal{B}) = \inf_{f \in \mathcal{B}} \left(\sup\{r \in (0, 1] : f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\}\right),$$

respectively.

We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \mathcal{A}$ $(|\omega(z)| \leq |z|, \quad z \in \mathcal{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence.

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Let γ, δ be real parameters, $0 \leq \gamma < 1$, $\delta \geq 0$, and let $\varphi, \phi \in \mathcal{A}_0(p, k)$. By $\mathcal{W}(p, k; \phi, \varphi; \gamma, \delta)$ we denote the class of functions $f \in \mathcal{A}(p, k)$ such that

$$(\varphi * f)(z) \neq 0 \ (z \in \mathcal{U} \setminus \{0\}) \tag{8}$$

and

$$\mathrm{Re} \left\{ \frac{(\phi * f)(z)}{(\varphi * f)(z)} - \gamma \right\} > \delta \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| \quad (z \in \mathcal{U}). \tag{9}$$

Also, let us denote

$$\begin{aligned}
\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta) \quad &: \quad = \mathcal{T}(p, k) \cap \mathcal{W}(p, k; \phi, \varphi; \gamma, \delta), \\
\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta) \quad &: \quad = \mathcal{T}(p, k; \eta) \cap \mathcal{W}(p, k; \phi, \varphi; \gamma, \delta), \\
\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta) \quad &: \quad = \mathcal{A}_\rho(p, k) \cap \mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta), \\
\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta) \quad &: \quad = \mathcal{A}_\rho(p, k) \cap \mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta).
\end{aligned}$$

For the presented investigations we assume that φ, ϕ are the functions of the form

$$\varphi(z) = z^p + \sum_{n=k}^{\infty} \alpha_n z^n, \quad \phi(z) = z^p + \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{U}), \tag{10}$$

where

$$0 \leq \alpha_n < \beta_n \quad (n = k, k+1, ...).$$

Moreover, let us put

$$d_n := (\delta + 1) \beta_n - (\delta + \gamma) \alpha_n \quad (n = k, k+1, ...). \tag{11}$$

The families $\mathcal{W}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right)$ and $\mathcal{W}_\rho\left(p, k; \phi, \varphi; \gamma, \delta\right)$ unify various new and well-known classes of analytic functions. In particular, the class

$$\mathcal{W}_\rho\left(\varphi; \gamma, \delta; \eta\right) := \mathcal{W}_\rho\left(p, k; \frac{z\varphi'\left(z\right)}{p}, \varphi\left(z\right); \gamma, \delta; \eta\right),$$

contains functions $f \in \mathcal{A}_\rho\left(p, k\right)$, such that

$$\mathrm{Re}\left\{\frac{z\left(\varphi * f\right)'\left(z\right)}{p\left(\varphi * f\right)\left(z\right)} - \gamma\right\} > \delta\left|\frac{z\left(\varphi * f\right)'\left(z\right)}{p\left(\varphi * f\right)\left(z\right)} - 1\right| \quad \left(z \in \mathcal{U}\right).$$

The class

$$\mathcal{H}_\mathcal{T}\left(\varphi; \gamma, \delta\right) := \mathcal{T}\mathcal{W}_0\left(1, 2; \varphi; \gamma, \delta; 0\right)$$

was introduced and studied by Raina and Bansal [4]. If we set

$$h(\alpha_1, z) := z\,_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where $_qF_s$ is the generalized hypergeometric function (see for details [5] and [6]), then we obtain the class

$$\mathcal{U}\mathcal{H}\left(q, s, \lambda, \gamma, \delta\right) := \mathcal{T}\mathcal{W}_0\left(1, 2; \lambda h(\alpha_1 + 1, z) + \left(1 - \lambda\right)h(\alpha_1, z); \gamma, \delta; 0\right) \quad \left(0 \leq \lambda \leq 1\right)$$

defined by Srivastava *et al.* [7]. The classes

$$\begin{aligned}
\delta - UST\left(\gamma\right) &= \mathcal{W}_0\left(1, 2; \frac{z}{1-z}; \gamma, \delta\right), \\
\delta - UCV\left(\gamma\right) &= \mathcal{W}_0\left(1, 2; \frac{z}{\left(1-z\right)^2}; \gamma, \delta\right),
\end{aligned}$$

are the well-known classes of of δ-starlike function of order γ and δ-uniformly convex function of order γ, respectively. In particular, the classes $UCV := UCV\left(1, 0\right)$, $\delta - UCV := UCV(\delta, 0)$ were introduced by Goodman [8] (see also [9, 10, 11]), and Kanas and Wisniowska [12], respectively.

Many other classes, are also particular cases of the class investigated here, see for example [13, 14, 15].

The object of the present paper is to investigate the coefficients estimates, distortion properties and the radii of starlikeness and convexity.

2   Coefficients estimates

We   first   mention a sufficient condition for the function to belong to the class $\mathcal{W}\left(p, k; \phi, \varphi; \gamma, \delta\right)$.

Theorem 1 *Let $\{d_n\}$ be defined by (11), and let $0 \leq \gamma < 1$. If a function f of the form (1) satisfies the condition*

$$\sum_{n=k}^{\infty} d_n\left|a_n\right| \leq \left(1 - \gamma\right)a_p, \tag{12}$$

then f belongs to the class $\mathcal{W}(p, k; \phi, \varphi; \gamma, \delta)$.

Proof. By definition of the class $\mathcal{W}(p,k;\phi,\varphi;\gamma,\delta)$, it suffices to show that

$$\delta\left|\frac{(\phi*f)(z)}{(\varphi*f)(z)}-1\right|-\operatorname{Re}\left\{\frac{(\phi*f)(z)}{(\varphi*f)(z)}-1\right\}\leq 1-\gamma \quad (z\in\mathcal{U}). \tag{13}$$

Simply calculations give

$$\delta\left|\frac{(\phi*f)(z)}{(\varphi*f)(z)}-1\right|-\operatorname{Re}\left\{\frac{(\phi*f)(z)}{(\varphi*f)(z)}-1\right\}$$

$$\leq (\delta+1)\left|\frac{(\phi*f)(z)}{(\varphi*f)(z)}-1\right|\leq (\delta+1)\frac{\sum\limits_{n=k}^{\infty}(\beta_n-\alpha_n)|a_n||z|^{n-p}}{a_p-\sum\limits_{n=k}^{\infty}\alpha_n|a_n||z|^{n-p}}.$$

Now the last expression is bounded above by $(1-\gamma)$ if (12) holds. Whence $f\in\mathcal{W}(p,k;\phi,\varphi;\gamma,\delta)$. $\blacksquare$

Our next theorem shows that the condition (12) is necessary as well for functions of the form (1), with (4) to belong to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

Theorem 2 *Let f be a function of the form (1), satisfying the argument property (4). Then f belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ if and only if the condition (12) holds true.*

Proof. In view of Theorem 1 we need only show that each function f from the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ satisfies the coefficient inequality (12). Let a function f of the form (1), satisfying the argument property (4) belong to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$. Then by (9), we have

$$\delta\left|\frac{a_pz^p+\sum\limits_{n=k}^{\infty}\beta_na_nz^n}{a_pz^p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^n}-1\right|<\operatorname{Re}\left\{\frac{a_pz^p+\sum\limits_{n=k}^{\infty}\beta_na_nz^n}{a_pz^p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^n}-\gamma\right\},$$

or equivalently

$$\delta\left|\frac{\sum\limits_{n=k}^{\infty}(\beta_n-\alpha_n)a_nz^{n-p}}{a_p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^{n-p}}\right|<\operatorname{Re}\left\{\frac{(1-\gamma)a_p+\sum\limits_{n=k}^{\infty}(\beta_n-\gamma\alpha_n)a_nz^{n-p}}{a_p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^{n-p}}\right\}.$$

In view of (4), we set $z=re^{i\eta}$ $(0\leq r<1)$ in the above inequality to obtain

$$\frac{\sum\limits_{n=k}^{\infty}\delta(\beta_n-\alpha_n)|a_n|r^{n-p}}{a_p-\sum\limits_{n=k}^{\infty}\alpha_n|a_n|r^{n-p}}<\frac{(1-\gamma)a_p-\sum\limits_{n=k}^{\infty}(\beta_n-\gamma\alpha_n)|a_n|r^{n-p}}{a_p-\sum\limits_{n=k}^{\infty}\alpha_n|a_n|r^{n-p}}.$$

Thus, by (8) we have

$$\sum_{n=k}^{\infty} \left[(\delta + 1) \beta_n - (\delta + \gamma) \alpha_n \right] |a_n| r^{n-p} < (1 - \gamma) a_p,$$

which, upon letting $r \to 1^-$, readily yields the assertion (12). ■

By applying Theorem 2 we can deduce following result.

Theorem 3 *Let f be a function of the form (1), satisfying the argument property (4). A function f of the form (1) belongs to the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)$ if and only if it satisfies (3) and*

$$\sum_{n=k}^{\infty} \left(d_n - (1 - \gamma) |\rho|^{n-p} \right) |a_n| \leq 1 - \gamma, \tag{14}$$

where $\{d_n\}$ is defined by (11).

Proof. For a function f of the form (1) with the normalization (3), we have

$$a_p = 1 + \sum_{n=k}^{\infty} |a_n| \, |\rho|^{n-p}. \tag{15}$$

Applying the equality (15) to (12), we obtain the assertions (14). ■

Since the condition (14) is independent of η, Theorem 3 yields the following theorem.

Theorem 4 *Let f be a function of the form (1), satisfying the argument property (4). Then f belongs to the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta)$ if and only if the condition (14) holds true.*

By applying Theorem 3 we obtain the following lemma.

Lemma 1 *Let $\{d_n\}$ be defined by (11), $\rho \in \mathcal{U}$, and let us assume, that there exists an integer n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, ...\})$ such that*

$$d_{n_0} - (1 - \gamma) |\rho|^{n_0 - p} \leqq 0. \tag{16}$$

Then the function

$$f_{n_0}(z) = \left(1 + a \rho^{n_0 - p} \right) z^p - a e^{i(p - n_0)\eta} z^{n_0}$$

belongs to the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)$ for all positive real numbers a. Moreover, for all n $(n \in \mathbb{N}_k)$ such that

$$d_n - (1 - \gamma) |\rho|^{n-p} > 0, \tag{17}$$

the functions

$$f_n(z) = \left(1 + a\rho^{n_0-p} + bz^{n-p}\right) z^p - ae^{i(p-n_0)\eta} z^{n_0} - be^{i(p-n)\eta} z^n,$$

where

$$b = \frac{1 - \gamma + \left((1-\gamma)\,|\rho|^{n_0-p} - d_{n_0}\right) a}{d_n - (1-\gamma)\,|\rho|^{n-p}},$$

belong to the class $\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right).$

By Lemma 1 and Theorem 3, we have following corollary.

Corollary 1 *Let a function f of the form (1) belongs to the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)$ and let $\{d_n\}$ be defined by (11). Then all of the coefficients a_n for which*

$$d_n - (1-\gamma)\,|\rho|^{n-p} = 0$$

are unbounded. Moreover, if there exists an integer n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, ...\})$ such that

$$d_{n_0} - (1-\gamma)\,|\rho|^{n_0-p} < 0,$$

then all of the coefficients of the function f are unbounded. In the remaining cases

$$|a_n| \leqq \frac{1-\gamma}{d_n - (1-\gamma)\,|\rho|^{n-p}}\ .$$

The result is sharp, the functions f_n of the form

$$f_{n,\eta}(z) = \frac{d_n z^p - (1-\gamma)\,e^{i(p-n)\eta} z^n}{d_n - (1-\gamma)\,|\rho|^{n-p}} \quad (z \in \mathcal{U};\ n = k, k+1, \ldots)$$

are the extremal functions.

Remark 1 *The coefficients estimates for the class $\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta\right)$ are the same as for the class $\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right)$.*

By puting $\rho = 0$ in Theorems 3 and 4, and Corollary 1, we have the corollaries listed below.

Corollary 2 *Let f be a function of the form (1), satisfying the argument property (4). A function f of the form (1) belongs to the class $\mathcal{TW}_0\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right)$ if and only if*

$$\sum_{n=k}^{\infty} d_n\,|a_n| \leq 1 - \gamma, \tag{18}$$

where $\{d_n\}$ is defined by (11).

Corollary 3 *Let f be a function of the form (1), satisfying the argument property (4). Then f belongs to the class $\mathcal{TW}_0\left(p,k;\phi,\varphi;\gamma,\delta\right)$ if and only if the condition (18) holds true.*

Corollary 4 *If a function f of the form (1) belongs to the class $\mathcal{TW}_0\left(p,k;\phi,\varphi;\gamma,\delta;\eta\right),$ then*

$$|a_n| \leq \frac{1-\gamma}{d_n} \quad (n=k,k+1,\ldots), \tag{19}$$

where d_n is defined by (11). The result is sharp. The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z) = z^p - \frac{1-\gamma}{d_n} e^{i(p-n)\eta} z^n \quad (z \in \mathcal{U}; \ n=k,k+1,\ldots) \tag{20}$$

are the extremal functions.

Corollary 5 *If a function f of the form (1) belongs to the class $\mathcal{TW}_0\left(p,k;\phi,\varphi;\gamma,\delta\right)$, then the coefficients estimates (19) holds true. The result is sharp. The functions $f_{n,\eta}$ $(\eta \in \mathbb{R})$ of the form (20) are the extremal functions.*

3   Distortion theorems

From Theorem 2 we have the following lemma.

Lemma 2 *Let a function f of the form (1) belong to the class $\mathcal{TW}_\rho\left(p,k;\phi,\varphi;\gamma,\delta;\eta\right)$. If the sequence $\{d_n\}$ defined by (11) satisfies the inequality*

$$0 < d_k - (1-\gamma)|\rho|^{k-p} \leq d_n - (1-\gamma)|\rho|^{n-p} \quad (n=k,k+1,\ldots), \tag{21}$$

then

$$\sum_{n=k}^{\infty} |a_n| \leq \frac{1-\gamma}{d_k - (1-\gamma)|\rho|^{k-p}}.$$

Moreover, if

$$0 < \frac{d_k - (1-\gamma)|\rho|^{k-p}}{k} \leq \frac{d_n - (1-\gamma)|\rho|^{n-p}}{n} \quad (n=k,k+1,\ldots), \tag{22}$$

then

$$\sum_{n=k}^{\infty} n|a_n| \leq \frac{k(1-\gamma)}{d_k - (1-\gamma)|\rho|^{k-p}}.$$

Remark 2 *The second part of Lemma 2 we can rewritten in terms of σ-neighborhood N_σ defined by*

$$N_\sigma = \left\{ f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \in \mathcal{T}(p,k;\eta) : \quad \sum_{n=k}^{\infty} n|a_n| \leq \sigma \right\}$$

in the following form:
if the sequence $\{d_n\}$ defined by (11) satisfies (22), then

$$\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right) \subset N_\sigma.$$

where

$$\delta = \frac{k\left(1 - \gamma\right)}{d_k - \left(1 - \gamma\right)\left|\rho\right|^{k-p}}.$$

Theorem 5 *Let a function f belong to the class $\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right)$ and let $|z| = r < 1$. If the sequence $\{d_n\}$ defined by (11) satisfies (21), then*

$$\phi(r) \leq |f(z)| \leq \frac{d_k r^p + \left(1 - \gamma\right) r^k}{d_k - \left(1 - \gamma\right)\left|\rho\right|^{k-p}}, \tag{23}$$

where

$$\phi(r) := \begin{cases} r^p & (r \leqq \rho) \\ \frac{d_k r^p - (1-\gamma) r^k}{d_k - (1-\gamma)|\rho|^{k-p}} & (r > \rho). \end{cases} \tag{24}$$

Moreover, if (22) holds, then

$$pa_p r^{p-1} - \frac{k\left(1 - \gamma\right)}{d_k - \left(1 - \gamma\right)\left|\rho\right|^{k-p}} r^{k-1} \leq |f'(z)| \leq \frac{pd_k r^p + k\left(1 - \gamma\right) r^{k-1}}{d_k - \left(1 - \gamma\right)\left|\rho\right|^{k-p}}. \tag{25}$$

The result is sharp, with the extremal function $f_{k,\eta}$ of the form (20) and $f(z) = z$.

Proof. Suppose that the function f of the form (1) belongs to the class $\mathcal{TW}_\rho\left(p, k; \phi, \varphi; \gamma, \delta; \eta\right)$. By Lemma 2 we have

$$\begin{aligned} |f(z)| &= \left|a_p z^p + \sum_{n=k}^{\infty} a_n z^n\right| \leq r^p \left(a_p + \sum_{n=k}^{\infty} |a_n| r^{n-p}\right) \\ &\leq r^p \left(1 + \sum_{n=k}^{\infty} |a_n| |\rho|^{n-p} + \sum_{n=k}^{\infty} |a_n| r^{n-p}\right) \\ &\leq r^p \left(1 + (|\rho|^{k-p} + r^{k-p}) \sum_{n=k}^{\infty} |a_n|\right) \leq \frac{d_k r^p + (1-\gamma) r^k}{d_k - (1-\gamma)|\rho|^{k-p}}, \end{aligned}$$

and

$$|f(z)| \geq r^p \left(a_p - \sum_{n=k}^{\infty} |a_n| r^{n-p}\right) = r^p \left(1 + \sum_{n=k}^{\infty} (|\rho|^{n-p} - r^{n-p}) |a_n|\right). \tag{26}$$

If $r \leqq \rho$, then we obtain $|f(z)| \geqq r^p$. If $r > \rho$, then the sequence $\left\{\left(\rho^{n-1} - r^{n-1}\right)\right\}$ is decreasing and negative. Thus, by (26), we obtain

$$|f(z)| \geq r^p \left(1 - (r^{k-p} - |\rho|^{k-p}) \sum_{n=2}^{\infty} a_n\right) \geqq \frac{d_k r^p - \left(1 - \gamma\right) r^k}{d_k - \left(1 - \gamma\right)\left|\rho\right|^{k-p}},$$

and we have the assertion (23). Making use of Lemma 2, in conjunction with (15), we readily obtain the assertion (25) of Theorem 5. ∎

Theorem 5 implies the following results.

Corollary 6 *Let a function f belong to the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta)$. If the sequence $\{d_n\}$ defined by (11) satisfies (21), then the assertion (23) holds true. Moreover, if we assume (22), then then the assertion (25) holds true. The result is sharp, with the extremal functions $f_{k,\eta}$ $(\eta \in \mathbb{R})$ of the form (20).*

Corollary 7 *Let a function f belong to the class $\mathcal{TW}_0(p, k; \phi, \varphi; \gamma, \delta; \eta)$ and let the sequence $\{d_n\}$ be defined by (11). If*

$$d_k \leq d_n \quad (n = k, k+1, \ldots), \tag{27}$$

then

$$r^p - \frac{1-\gamma}{d_k} r^k \leq |f(z)| \leq r^p + \frac{1-\gamma}{d_k} r^k \quad (|z| = r < 1). \tag{28}$$

Moreover, if

$$nd_k \leq kd_n \quad (n = k, k+1, \ldots), \tag{29}$$

then

$$pr^{p-1} - \frac{k(1-\gamma)}{d_k} r^{k-1} \leq |f'(z)| \leq pr^{p-1} + \frac{k(1-\gamma)}{d_k} r^{k-1} \quad (|z| = r < 1). \tag{30}$$

The result is sharp, with the extremal function $f_{k,\eta}$ of the form (20).

Corollary 8 *Let a function f belong to the class $\mathcal{TW}_0(p, k; \phi, \varphi; \gamma, \delta)$. If the sequence $\{d_n\}$ defined by (11) satisfies (27), then the assertion (28) holds true. Moreover, if we assume (29), then then the assertion (28) holds true. The result is sharp, with the extremal functions $f_{k,\eta}$ $(\eta \in \mathbb{R})$ of the form (20).*

4   The Radii of convexity and starlikeness

Theorem 6 *The radius of starlikeness of order α for the class $\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta)$ is given by*

$$R_\alpha^*(\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta)) = \inf_{n \geq k} \left(\frac{(p-\alpha) d_n}{(n-\alpha)(1-\gamma)} \right)^{\frac{1}{n-p}}, \tag{31}$$

where d_n is defined by (11). The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z) = a_p \left(z^p - \frac{1-\gamma}{d_n} e^{i(p-n)\eta} z^n \right) \quad (z \in \mathcal{U}; \; n = k, k+1, \ldots; a_p > 0) \tag{32}$$

are the extremal functions.

Proof.   A function $f \in \mathcal{T}(p, k; \eta)$ of the form (1) is starlike of order α in the disk $\mathcal{U}(r)$, $0 < r \leq 1$, if and only if it satisfies the condition (7). Since

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{\sum_{n=k}^{\infty} (n-p) a_n z^n}{a_p z^p + \sum_{n=k}^{\infty} a_n z^n} \right| \leq \frac{\sum_{n=k}^{\infty} (n-p) |a_n| |z|^{n-p}}{a_p - \sum_{n=k}^{\infty} |a_n| |z|^{n-p}},$$

putting $|z| = r$ the condition (7) is true if

$$\sum_{n=k}^{\infty} \frac{n-\alpha}{p-\alpha} |a_n| r^{n-p} \leq a_p. \tag{33}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty} \frac{d_n}{1-\gamma} |a_n| \leq a_p, \tag{34}$$

Thus, the condition (33) is true if

$$\frac{n-\alpha}{p-\alpha} r^{n-p} \leq \frac{d_n}{1-\gamma} \qquad (n = k, k+1, \ldots),$$

that is, if

$$r \leq \left(\frac{(p-\alpha) d_n}{(n-\alpha)(1-\gamma)} \right)^{\frac{1}{n-p}} \qquad (n = k, k+1, \ldots). \tag{35}$$

It follows that each function $f \in \mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta)$ is starlike of order α in the disk $\mathcal{U}(r)$, where

$$r = \inf_{n \geq k} \left(\frac{(p-\alpha) d_n}{(n-\alpha)(1-\gamma)} \right)^{\frac{1}{n-p}}$$

The functions $f_{n,\eta}$ of the form (32) realize equality in (34), and the radius r can not be larger. Thus we have (31). ∎

The following result may be proved in much the same way as Theorem 6.

Theorem 7 *The radius of convexity of order α for the class $\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta)$ is given by*

$$R_\alpha^c (\mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta)) = \inf_{n \geq k} \left(\frac{(p-\alpha) d_n}{n(n-\alpha)(1-\gamma)} \right)^{\frac{1}{n-p}},$$

where d_n is defined by (11). The functions $f_{n,\eta}$ of the form (32) are the extremal functions.

It is clear that for

$$a_p = \frac{d_n}{d_n - (1-\gamma)|\rho|^{n-p}} > 0$$

the extremal functions $f_{n,\eta}$ of the form (32) belong to the class $\mathcal{TW}_\rho (p, k; \phi, \varphi; \gamma, \delta; \eta)$. Moreover, we have

$$\mathcal{TW}_\rho (p, k; \phi, \varphi; \gamma, \delta; \eta) \subset \mathcal{TW}(p, k; \phi, \varphi; \gamma, \delta; \eta).$$

Thus, by Theorems 6 and 7 we have the following results.

Corollary 9 *Let the sequence $\left\{d_n - (1 - \gamma)|\rho|^{n-p}\right\}$, where d_n is defined by (11), be positive. The radius of starlikeness of order α for the class $TW_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)$ is given by*

$$R_\alpha^* \left(\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)\right) = \inf_{n \geq k} \left(\frac{(p - \alpha) d_n}{(n - \alpha)(1 - \gamma)}\right)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (32) are the extremal functions.

Corollary 10 *Let the sequence $\left\{d_n - (1 - \gamma)|\rho|^{n-p}\right\}$, where d_n is defined by (11), be positive. The radius of convexity of order α for the class $\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)$ is given by*

$$R_\alpha^c \left(\mathcal{TW}_\rho(p, k; \phi, \varphi; \gamma, \delta; \eta)\right) = \inf_{n \geq k} \left(\frac{(p - \alpha) d_n}{n(n - \alpha)(1 - \gamma)}\right)^{\frac{1}{n-p}},$$

where d_n is defined by (11).

Remark 3 *We conclude this paper by observing that, in view of the definitions of investigated classes which is expressed in terms of the convolution of the functions in (10), involving arbitrary sequences, our main results can lead to several additional new results. In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of analytic functions which would incorporate linear operators. Some of these results were obtained in earlier works, see for example [16, 17, 18, 19, 20].*

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Monotonic continuous solution for a mixed type integral inclusion of fractional order

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ABSTRACT: In this paper we are concerned with the mixed type integral inclusion

$$x(t) \in p(t) + \int_0^1 k(t,s) \; F_1(s, I^\beta f_2(s, x(s))) ds, \quad t \in [0,1].$$

The existence of monotonic continuous solution will be proved. As an application the initial-value problem of the arbitrary (fractional) orders differential inclusion

$$\frac{dx(t)}{dt} \in p(t) + \int_0^1 k(t,s) F_1(s, D^\alpha x(s)) ds, \quad a.e., \;\; t > 0$$

will be studied.

AMS Subject Classification:
Key Words and Phrases: *Fractional calculus; Caratheodory condition; fixed point theorem; mixed type integral inclusion.*

1 Introduction

The existence of monotonic integrable solution for the mixed type nonlinear integral equation

$$x(t) = p(t) + \int_0^1 k(t,s) \; f_1(s, I^\beta \; f_2(s, x(s))) ds, \quad t \in [0,1], \;\; \beta > 0 \qquad (1)$$

has been studied in [6] where the given function P is nondecreasing on $[0,1]$ and the two functions f_1 and f_2 are monotonic nondecreasing (in both variables) and satisfy Caratheodory condition.

Here we relax the condition of monotonicity on the two functions f_1 and f_2 and prove the existence of positive continuous solution of (1).

When the given function p is nondecreasing and the kernel $k(t,s)$ is nondecreasing in t, $t \in [0,1]$, we prove that the solution of (1) is nondecreasing.

As a generalization of our results we study the existence of positive monotonic continuous solution of the mixed type integral inclusion

$$x(t) \in p(t) + \int_0^1 k(t,s)F_1(s, I^\beta f_2(s, x(s)))ds, \quad t \in [0,1], \quad \beta > 0 \qquad (2)$$

where the set-valued map $F(t,.)$ is lower semicontinuous from R^+ into R^+ and $F(.,.)$ is measurable.

Finally the differential inclusion of arbitrary (fractional) orders

$$\frac{dx(t)}{dt} \in p(t) + \int_0^1 k(t,s)F_1(s, D^\alpha x(s))ds, \quad a.e., \quad t > 0 \qquad (3)$$

with the initial data

$$x(0) = x_\circ \quad \geq 0 \qquad (4)$$

will be studied.

2 Preliminaries

Let $L^1(I)$ be the class of Lebesgue integrable functions defined on the interval $I = [a,b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function.

Definition 2.1 The fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by ([7], [9] and [12])

$$I_a^\alpha \ f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s) \ ds.$$

Definition 2.2 The (Caputo) fractional derivative D^α of order $\alpha \in (0,1]$ of the absolutely continuous function g is defined as ([2], [9], [10] and [12])

$$D_a^\alpha \ g(t) = I_a^{1-\alpha} \ \frac{d}{dt} \ g(t) \ , \quad t \in [a,b].$$

Now, we shall state the following theorems which are used in the sequel.

Theorem 2.1 Schauder's fixed-point Theorem [8]
Let S be a convex subset of a Banach space B, let the mapping $T : S \to S$ be compact and continuous. Then T has at least one fixed-point in S.

Theorem 2.2 Arzela -Ascoli Theorem [4]

Let E be a compact metric space and $C(E)$ be the Banach space of real or complex valued continuous function normed by

$$\|f\| = \sup_{t \in E} |f(t)|.$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equicontinuous, then $\bar{A}$ is compact.

3 Main results

Let $C(I)$, $I = [0, 1]$ be the class of continuous functions defined on I.

In this section we present our main result by proving the existence of monotonic positive solution $x \in C(I)$ for the mixed type integral equation (1).

To facilitate our discussion, let us first state the following assumptions:

(i) $p : [0, 1] \to R^+$ is continuous. There is a positive constant p such that $|p(t)| < p$.

(ii) $f_i : [0, 1] \times R^+ \to R^+$, $i = 1, 2$ satisfy caratheodory condition i.e. f is measurable in t for any $x \in R^+$ and continuous in x for almost all $t \in [0, 1]$.
There exist two functions $a_1, a_2 \in L^1$ and two positive numbers b_1, b_2 such that

$$|f_i(t, x)| \le a_i(t) + b_i|x|, \quad i = 1, 2 \ \forall \ t \in [0, 1] \quad \text{and} \quad x \in R^+.$$

(iii) $k : [0, 1] \times [0, 1] \to R^+$ is continuous in t for any $s \in [0, 1]$ and measurable in s for any $t \in [0, 1]$ such that

$$\int_0^1 k(t, s)(a_1(s) + b_1 I^\beta a_2(s)) \ ds \le M_1 \quad \text{and} \quad \int_0^1 k(t, s)s^\beta ds < M_2.$$

(iv) $b_1 \, b_2 \, M_2 < \Gamma(\beta + 1)$.

Remark: It must be noticed that assumption (iii) implies that the two functions

$$\int_0^1 k(t, s)(a_1(s) + b_1 I^\beta a_2(s)) \ ds \quad \text{and} \quad \int_0^1 k(t, s)s^\beta ds.$$

are continuous in t, $t \in [0, 1]$.

Now, we are in position to formulate and prove our main result.

Theorem 3.1 Let the assumptions (i)-(iv) be satisfied. Then equation (1) has at least one positive solution $x \in C(I)$.

Proof Define the subset S of $C(I)$ by

$$S = \{ \, x \in C \, : \, |x(t)| \leq \, r \, \}, \qquad t \in [0, 1],$$

where r is a positive constant. It is clear that S is closed and convex.
Let T be an operator defined by

$$(Tx)(t) = p(t) + \int_0^1 k(t, s) f_1(s, I^\beta f_2(s, x(s))) ds \quad \forall \, x \in S. \tag{5}$$

Assumption (ii) implies that $T : S \to C$ is continuous in x.
Now for every $x \in S$ we have

$$|(Tx)(t)| \leq |p(t)| + \int_0^1 k(t, s) \, |f_1(s, I^\beta \, f_2(s, x(s)))| ds$$

$$\leq \, p \, + \, \int_0^1 k(t, s) \, [a_1(s) \, + \, b_1 I^\beta f_2(s, x(s))|] ds$$

$$\leq \, p \, + \, \int_0^1 k(t, s) a_1(s) ds \, + \, b_1 \int_0^1 k(t, s) I^\beta [a_2(s) + b_2 |x(s)|] ds$$

$$\leq \, p \, + \, \int_0^1 k(t, s) [a_1(s) \, + \, b_1 I^\beta a_2(s)] ds \, + \, b_1 b_2 \int_0^1 k(t, s) \, I^\beta |x(s)| ds$$

$$\leq \, p \, + \, M_1 \, + \, \frac{b_1 b_2 r}{\Gamma(\beta + 1)} \int_0^1 k(t, s) s^\beta ds$$

$$\leq \, p \, + \, M_1 \, + \, \frac{b_1 b_2 \, M_2 \, r}{\Gamma(\beta + 1)}.$$

Therefore,

$$|(Tx)(t)| \, \leq \, p \, + \, M_1 \, + \, \frac{b_1 b_2 \, M_2 \, r}{\Gamma(\beta + 1)}. \tag{6}$$

From the last estimate we deduce that

$$r \, = \, (p \, + \, M_1) \left(1 - \frac{b_1 b_2 M_2}{\Gamma(\beta + 1)} \right)^{-1}$$

and $Tx \in S$ and hence $TS \subset S$.
Also for $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$, we have

$$(Tx)(t_2) - (Tx)(t_1) = p(t_2) - p(t_1) + \int_0^1 (k(t_2, s) - k(t_1, s)) f_1(s, I^\beta f_2(s, x(s))) ds.$$

Then

$$|(Tx)(t_2) - (Tx)(t_1)| \leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| f_1(s, I^\beta f_2(s, x(s)))| ds$$

$$\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| [a_1(s) + b_1| I^\beta f_2(s, x(s))|] ds$$

$$\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| a_1(s) ds$$

$$+ b_1 \int_0^1 |k(t_2, s) - k(t_1, s)| |I^\beta f_2(s, x(s))| ds$$

$$\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| a_1(s) ds + b_1 \int_0^1 |k(t_2, s) - k(t_1, s)| I^\beta a_2(s) ds$$

$$+ b_1 b_2 \int_0^1 |k(t_2, s) - k(t_1, s)| I^\beta |x(s)| ds$$

$$\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| [a_1(s) + b_1 I^\beta a_2(s)] ds$$

$$+ b_1 b_2 r \int_0^1 |k(t_2, s) - k(t_1, s)| \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)} d\tau ds$$

$$\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2, s) - k(t_1, s)| [a_1(s) + b_1 I^\beta a_2(s)] ds$$

$$+ \frac{b_1 b_2 r}{\Gamma(\beta + 1)} \int_0^1 |k(t_2, s) - k(t_1, s)| s^\beta ds.$$

From the last inequality, the continuity of the function p and assumption (iii) we deduce the equicontinuity of the functions of TS on $[0, 1]$. Then by Arzela-Ascoli Theorem the closure of TS is compact.

Now, all conditions of Schauder's fixed-point Theorem are hold, then T has a fixed point in S. Hence there exists at least one positive solution $x \in C(I)$ of (1).

Corollary 3.1 Let the assumption (i)-(iv) are satisfied. If the function p is nondecreasing and k is nondecreasing in $t \in I$, then the solution of (1) is nondecreasing.

Proof   For $t_1, t_2 \in I$ and $t_1 < t_2$, we have

$$x(t_1) = p(t_1) + \int_0^1 k(t_1, s) \ f_1(s, I^\beta f_2(s, x(s))) \ ds$$

$$\leq \ p(t_2) + \int_0^1 k(t_2, s) \ f_1(s, I^\beta f_2(s, x(s))) \ ds = x(t_2).$$

4   Mixed type integral inclusion

Consider now the integral inclusion (2), where $F_1 : [0, 1] \times R^+ \to 2^{R^+}$ has nonempty closed convex values.

As an important consequence of the main result we can present the following:

Theorem 4.1   Let the assumptions of Theorem 3.1 are satisfied and the multifunction F_1 satisfies the following assumptions:

(1) $F_1(t, x)$ are non empty, closed and convex for all $(t, x) \in [0, 1] \times R^+$,

(2) $F_1(t, .)$ is lower semicontinuous from R^+ into R^+,

(3) $F_1(., .)$ is measurable,

(4) There exist a function $a_1 \in L^1$ and a positive number b_1 such that

$$|F_1(t, x)| \ \leq \ a_1(t) \ + \ b_1 \ |x| \quad \forall \ t \in [0, 1].$$

Then there exists at least one positive solution $x \in C(I)$ of the integral inclusion (2).

Proof   By conditions $(1) - (4)$ (see [1], [3], [5] and [11]) we can find a selection function f_1 (Caratheodory function) $f_1 : [0, 1] \times R^+ \to R^+$ such that $f_1(t, x) \in F_1(t, x)$ for all $(t, x) \in [0, 1] \times R^+$, this function satisfies condition (ii) of Theorem 3.1.

Clearly all assumption of Theorem 3.1 are hold, then there exists a continuous positive solution $x \in C(I)$ such that

$$x(t) - p(t) = \int_0^1 K(t, s) \ f_1(s, I^\beta \ f_2(s, x(s))) ds \ \in \ \int_0^1 K(t, s) \ F_1(s, I^\beta \ f_2(s, x(s))) ds.$$

Now, we can easily prove the following Corollary.

Corollary 4.1 Let the assumptions of Theorem 4.1 and the Corollary 3.2 are satisfied, then the solution of (1) is nondecreasing.

5   Differential inclusion

Consider now the initial value problem of the differential inclusion (3) with the initial data (4).

Theorem 5.1 Let the assumptions of Theorem 4.1 are satisfied, then the initial value problem (3)-(4) has at least one positive nondecreasing solution $x \in C(I)$.
Proof   Let $y(t) = \frac{dx(t)}{dt}$, then equation (3) transformed to the integral inclusion

$$y(t) \in p(t) + \int_0^1 k(t,s)F_1(s, I^{1-\alpha}y(s))ds$$

which by Theorem 4.1 has at least one positive solution $y \in C(I)$.
This implies that the existence of the absolutely continuous solution

$$x(t) = x_\circ + \int_0^t y(s)ds$$

which is nondecreasing solution of the initial-value problem (3)-(4).

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On composition operator in the algebra of functions of two variables with bounded total Φ-variation in Schramm sense

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ABSTRACT: In this paper we consider the so called composition operator being a self-mapping of the Banach algebra of the function of two variables with bounded total Φ-variation in the Schramm sense. The main result of the paper characterizes the composition operator mentioned above which has a generating function being Lipschitzian with respect to the second variable. The basic tool used in our considerations is the concept of the left-left regularization.

1 Introduction

The nonlinear composition operator (which is also known as the superposition operator) is frequently used in many branches of nonlinear analysis and its applications. In order to define such an operator let us assume that I is a real interval (bounded or not) and $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ is a given function. For an arbitrary function $x(t) = x : I \to \mathbb{R}$ we may assign the function Fx defined as $(Fx)(t) = f(t,x(t))$ for $t \in I$. The operator F defined in such a way is called the composition operator generated by the function $f(t,x)$.

One of the basic problems considered in the theory of composition operator can be formulated as follows. Let us assume that $S(I)$ is a set (a space, an algebra, etc.) of some functions acting from I into $\mathbb{R}$. One has to formulate assumptions on the

function $f(t, x)$ guaranteeing that the composition operator F generated by $f(t, x)$ transforms $S(I)$ into itself.

Such a problem was solved in a lot of particular cases. We refer to the monograph [1] for more details concerning that problem.

The second important problem concerning the composition operator depends on the characterization of operators being Lipschitzian in a suitable function space. Such a problem in various situations was studied in a lot of papers (for instance [1], [2], [3], [4], [6], [7], [8], [11]).

In the paper we investigate the problem of characterization of the composition operator being a self-mapping of the Banach algebra of functions of two variables with bounded total variation in the Schramm sense. Namely, we show that such a composition operator is Lipschitzian if and only if it is affine.

The results obtained in the paper generalize those obtained for example in the papers [4], [10].

2   Preliminaries

In this section we collect all auxiliary facts which will be needed in the sequel. Let be $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be φ-function if it is continuous on $\mathbb{R}_+$, $\varphi(0) = 0$, φ is increasing on $\mathbb{R}_+$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Further, let $\Phi = \{\phi_n\}$ be a sequence of φ-functions. The sequence Φ is called the Φ-sequence if ϕ_n is convex and $\phi_{n+1}(t) \le \phi_n(t)$ for $n = 1, 2, \dots$ and for $t \in \mathbb{R}$. Apart from this we assume that these series $\sum \phi_n$ diverge for each $t > 0$.

Next, let us fix an interval $I = [a, b]$. Assume that $u : [a, b] \to \mathbb{R}$ is a given function. Let ϕ_n be a Φ-sequence of functions. If $I_n = [a_n, b_n]$ is a subinterval of the interval I we write $u(I_n) = u(b_n) - u(a_n)$ (for $n = 1, 2, \dots$).

We say that the function u has the *bounded total Φ-variation in the Schramm sense* on the interval $[a, b]$ if

$$\sum_n \phi_n\left(|u(I_n)|\right) < \infty$$

for each sequence $\{I_n\}$ of closed subintervals of I such that the intersection $I_i \cap I_j$ is empty or is a singleton for all $i, j = 1, 2, \dots, i \ne j$.

We introduced the $\Phi = \{\phi_{n,m}\}$ bidimensional sequence of increasing convex functions, such that $\phi_{n,m}(0) = 0$ and $\phi_{n,m}(t) > 0$ for $t > 0$ and $n, m = 1, 2, \dots$. We shall say that Φ is a Φ^*-sequence if $\phi_{n',m'}(t) \le \phi_{n,m}(t)$ for each $n' \le n$, $m' \le m$ and $t \in [0, \infty)$. If $\sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} \phi_{n,m}$ diverge for $t > 0$, we will say that Φ is a Φ-sequence.

In what follows let us assume that $a = (a_1, c_1)$, $b = (b_1, d_1)$ are two fixed points in the plane $\mathbb{R}^2$. Denote by I_a^b the rectangle generated by the points a and b, i.e. $I_a^b = [a_1, b_1] \times [c_1, d_1]$.

Further, let us take two sequences $\{I_n\}$, $\{J_m\}$ of the closed subintervals of the intervals $[a_1, b_1]$ and $[c_1, d_1]$, respectively. In other words, $I_n = [a_n, b_n]$ ($n = 1, 2, \dots$), $J_m =$

$[c_m, d_m]$ $(m = 1, 2, \ldots)$. Moreover, let $\Phi = \{\phi_{n,m}\}$ be a fixed double Φ-sequence and let $u : I_1 \to \mathbb{R}$ be a given function. Now, fix $x_2 \in J_1 = [c_1, d_1]$ and consider the function $u(\cdot, x_2) : I_1 \to \mathbb{R}$. The quantity V^S_{Φ, I_1} defined by the formula

$$
\begin{aligned}
V^S_{\Phi, I_1}(u) &= \sup \sum_{n=1}^{\infty} \phi_{n,m}\left(|u(I_n, x_2)|\right) \\
&= \sup \sum_{n=1}^{\infty} \phi_{n,m}\left(|u(b_n, x_2) - u(a_n, x_2)|\right)
\end{aligned} \tag{1}
$$

is called the Φ-variation in the Schramm sense of the function $u(\cdot, x_2)$. In the case when $V^S_{\Phi, I_1}(u) < \infty$ we will say that u has a *bounded Φ-variation in the sense of Schramm with respect to the first variable* (with the second one fixed).

In the same way we define the Φ-variation of the function $u(x_1, \cdot)$ in the Schramm sense, which will be denoted by V^S_{Φ, J_1}. If $V^S_{\Phi, J_1}(u) < \infty$ then u is said to be a function with *bounded Φ-variation in Schramm sense with respect to the second variable* (with the first one fixed).

Additionally, let us explain that the supremum in formula (1) is taken with respect to all sequences $\{I_n\}$ of the closed subintervals of the interval I_1. Obviously, in a similar way we understand the supremum in the formula of the quantity V^S_{Φ, J_1}.

Below we provide the definition of the main concept introduced in [4].

Definition 2.1. The quantity V^S_{Φ, I^b_a} defined by the formula

$$
\begin{aligned}
V^S_{\Phi, I^b_a}(u) &= \sup \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|u(I_n, J_m)|\right) \\
&= \sup \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|u(b_n, J_m) - u(a_n, J_m)|\right) \\
&= \sup \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|u(a_n, c_m) + u(b_n, d_m) - u(a_n, d_m) - u(b_n, c_m)|\right)
\end{aligned}
$$

is said to be the bidimensional variation in the sense of Schramm of the function u.

Now, let us set the quantity TV^S_Φ by putting

$$
TV^S_\Phi(u) = V^S_{\Phi, I_1}(u) + V^S_{\Phi, J_1}(u) + V^S_{\Phi, I^b_a}(u). \tag{2}
$$

This quantity is referred to the total Φ-variation of the function u in the Schramm sense. In the case when $TV^S_\Phi < \infty$ we say that u is a function with bounded total Φ-variation in Schramm sense.

The set of all functions $u : I^b_a \to \mathbb{R}$ having a bounded total Φ-variation will be denoted by $BV^S_\Phi(I^b_a)$.

Next, let us consider the functional P_Φ defined on the set $BV^S_\Phi(I^b_a)$ by the formula

$$
P_\Phi(f) = \inf\left\{\epsilon > 0 : TV^S_\Phi\left(\frac{f}{\epsilon}\right) \leq 1\right\}. \tag{3}
$$

The main result proved in [4] asserts that the set $BV_\Phi^S(I_a^b)$ forms a Banach algebra with the norm obtained by the formula

$$\|f\|_\Phi^S = |f(a)| + P_\Phi(f). \tag{4}$$

Our next result depends on the following lemma.

Lemma 2.1. *Let $f \in BV_\Phi^S(I_a^b)$ and $\Phi \in \Phi^*$. Then f has the following properties:*

(a) *If $(t,s), (t',s') \in I_a^b$ then $|f(t,s) - f(t',s')| \le 4\phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_\Phi(f)$.*

(b) *If $P_\Phi(f) > 0$ then $TV_\Phi^S(f/P_\Phi(f)) \le 1$.*

(c) *Let $r > 0$. Then $TV_\Phi^S(f/r) \le 1$ if and only if $P_\Phi(f) \le r$.*

(d) *If $r > 0$ and $TV_\Phi^S(f/P_\Phi(f)) = 1$ then $P_\Phi(f) = r$.*

In what follows let us fix arbitrary $f \in BV_\Phi^S(I_a^b)$. Then, the function $f^* : I_a^b \to \mathbb{R}$ defined by formula

$$f^*(x_1, x_2) := \begin{cases} \lim\limits_{\substack{y_1 \to x_1 - 0 \\ y_2 \to x_2 - 0}} f(y_1, y_2) & \text{if } (x_1, x_2) \in (a_1, b_1] \times (c_1, d_1], \\ \lim\limits_{\substack{y_1 \to x_1 - 0 \\ y_2 \to c_1 + 0}} f(y_1, y_2) & \text{if } x_1 \in (a_1, b_1] \text{ and } x_2 = c_1, \\ \lim\limits_{\substack{y_1 \to a_1 + 0 \\ y_2 \to x_2 - 0}} f(y_1, y_2) & \text{if } x_1 = a_1 \text{ and } x_2 \in (c_1, d_1], \\ \lim\limits_{\substack{y_1 \to a_1 + 0 \\ y_2 \to c_1 + 0}} f(y_1, y_2) & \text{if } x_1 = a_1 \text{ and } x_2 = c_1 \end{cases}$$

will be called the left-left regularization of the function f.
The existence of all one-side limits used above was proved in the book [5].

In the sequel we will denote by $G^-(I_a^b)$ the class of all left-left regularizations of the function $f \in BV_\Phi^S(I_a^b)$. It can be shown that $G^-(I_a^b)$ forms a linear space ([9]). Apart from this space $G^-(I_a^b)$ has the structure of a Banach space with respect to the norm

$$\|f\| = \sup\{|f| : (x,y) \in I_a^b\}.$$

To present the first result of this paper let us denote by $BV_{\Phi,*}^S(I_a^b)$ the subspace of the space $BV_\Phi^S(I_a^b)$ containing all functions being left-left continuous on $(a_1, b_1] \times (c_1, d_1]$.
We have the following result.

Lemma 2.2. *If $f \in BV_\Phi^S(I_a^b)$ then $f^* \in BV_{\Phi,*}^S(I_a^b)$.*

Proof. First, let us note that according to the definition of the left-left regularization, if $f \in BV_\Phi^S(I_a^b)$ then the function f^* is left-left continuous on the set $(a_1, b_1] \times (c_1, d_1]$. We show that $f^* \in BV_{\Phi,*}^S(I_a^b)$, i.e.

$$TV_\Phi^S(f^*) = V_{\Phi, I_1}^S(f^*) + V_{\Phi, J_1}^S(f^*) + V_{\Phi, I_a^b}^S(f^*) < \infty.$$

(cf. formulas (1) and (2)).

At the beginning we show that $V_{\Phi, I_1}^S(f^*) < \infty$.
To do this we fix $\epsilon > 0$ and take a partition π of the interval I_1 generated by the points $a_1 = t_0 < t_1 < \cdots < t_n = b_1$. Then by virtue of the definition of f^* we can find $t_i' \in (t_{i-1}, t_i) \subset [t_{i-1}, t_i] = \hat{I}_i$ $(i = 1, 2, \ldots, n)$ and $t_0' \in (a_1, t_1')$, $s_0 \in (c_1, d_1) \subset J_1$ such that

$$|f^*(t_i, c_1) - f^*(t_{i-1}, c_1)| \le |f(t_i', s_0) - f(t_{i-1}', s_0)| + \frac{1}{4}\phi_{n,m}^{-1}\left(\frac{\epsilon}{m}\right).$$

Hence, keeping in mind that $\phi_{n,m}$ is increasing, we deduce the following estimate

$$\begin{aligned}
\phi_{n,m}&(|f^*(t_i, c_1) - f^*(t_{i-1}, c_1)|) \\
&\le \phi_{n,m}\left(|f(t_i', s_0) - f(t_{i-1}', s_0)| + \frac{1}{4}\phi_{n,m}^{-1}\left(\frac{\epsilon}{m}\right)\right) \\
&= \phi_{n,m}\left(2\left[\frac{|f(t_i', s_0) - f(t_{i-1}', s_0)|}{2} + \frac{1}{2}\left(\frac{\phi_{n,m}^{-1}\left(\frac{\epsilon}{m}\right)}{4}\right)\right]\right) \\
&\le \frac{1}{2}\phi_{n,m}\left(2|f(t_i', s_0) - f(t_{i-1}', s_0)|\right) + \frac{1}{2}\phi_{n,m}\left(\frac{1}{2}\phi_{n,m}^{-1}\left(\frac{\epsilon}{m}\right)\right), \qquad (5)
\end{aligned}$$

which is also a consequence of the convexity of $\phi_{n,m}$.
On the other hand, since $\phi_{n,m}^{-1}$ is concave, we have

$$\begin{aligned}
\phi_{n,m}^{-1}\left(\frac{\epsilon}{m}\right) &= \phi_{n,m}^{-1}\left(2 \cdot \left(\frac{\epsilon}{2m}\right)\right) \\
&\le 2\phi_{n,m}^{-1}\left(\frac{\epsilon}{2m}\right). \qquad (6)
\end{aligned}$$

From (5) i (6) we obtain

$$\begin{aligned}
\phi_{n,m}&(|f^*(t_i, c_1) - f^*(t_{i-1}, c_1)|) \\
&\le \frac{1}{2}\phi_{n,m}\left(2|f(t_i', s_0) - f(t_{i-1}', s_0)|\right) + \frac{1}{2}\phi_{n,m}\left(\phi_{n,m}^{-1}\left(\frac{\epsilon}{2m}\right)\right) \\
&\le \phi_{n,m}\left(2|f(t_i', s_0) - f(t_{i-1}', s_0)|\right) + \phi_{n,m}\left(\phi_{n,m}^{-1}\left(\frac{\epsilon}{2m}\right)\right) \\
&= \phi_{n,m}\left(2|f(t_i', s_0) - f(t_{i-1}', s_0)|\right) + \frac{1}{2}\frac{\epsilon}{m}.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
\sum_{n=1}^k \phi_{n,m}\left(\left|f^*(\hat{I}_i, c_1)\right|\right) &\le \sum_{n=1}^k \phi_{n,m}\left(2|f(t_i', s_0) - f(t_{i-1}', s_0)|\right) + \epsilon \\
&\le V_{\Phi, I_1}^S(2f(\cdot, s_0)) + \epsilon \\
&\le V_{\Phi, I_1}^S(2f(\cdot, d_1)) + \epsilon,
\end{aligned}$$

since $c_1 < s_0 < d_1$. The last estimate allows us to derive the following one

$$V_{\Phi, I_1}^S(f^*(\cdot, c_1)) \le V_{\Phi, I_1}^S(2f(\cdot, d_1)) < \infty. \qquad (7)$$

Analogically we can show that

$$V_{\Phi,J_1}^S(f^*(a_1,\cdot)) \le V_{\Phi,J_1}^S(2f(b_1,\cdot)) < \infty. \tag{8}$$

In what follows fix two partitions π_1, π_2 of the intervals I_1, J_1, respectively, i.e. $\pi_1 : a_1 = t_0 < t_1 < \cdots < t_n = b_1$, $\pi_2 : c_1 = s_0 < s_1 < \cdots < s_m = d_1$. In view of the definition of f^* we infer that there exist $t_i' \in (t_{i-1}, t_i) \subset [t_{i-1}, t_i] = \hat{I}_i$ $(i = 1, 2, \ldots, n)$ and $s_j' \in (s_{j-1}, s_j) \subset [s_{j-1}, s_j] = \hat{J}_j$ $(j = 1, 2, \ldots, m)$, $t_0' \in (a_1, t_1')$, $s_0' \in (c_1, s_1')$ such that

$$\left| f^*(\hat{I}_i, \hat{J}_j) \right| = |f^*(t_{i-1}, s_{j-1}) + f^*(t_i, s_j) - f^*(t_{i-1}, s_j) - f^*(t_i, s_{j-1})|$$

$$\le \left| f(t_{i-1}', s_{j-1}') + f(t_i', s_j') - f(t_{i-1}', s_j') - f(t_i', s_{j-1}') \right| + \frac{1}{4}\phi_{n,m}^{-1}\left(\frac{\epsilon}{nm}\right).$$

In a similar way, as earlier, we obtain

$$\phi_{n,m}\left(\left| f^*(\hat{I}_i, \hat{J}_j) \right|\right)$$
$$\le \phi_{n,m}\left(2 \left| f(t_{i-1}', s_{j-1}') + f(t_i', s_j') - f(t_{i-1}', s_j') - f(t_i', s_{j-1}') \right|\right) + \frac{\epsilon}{nm}.$$

This yields

$$\sum_{n=1}^k \sum_{m=1}^l \phi_{n,m}\left(\left| f^*(\hat{I}_i, \hat{J}_j) \right|\right) \le \sum_{n=1}^k \sum_{m=1}^l \phi_{n,m}\left(2| f(t_{i-1}', s_{j-1}') + f(t_i', s_j') \right.$$
$$\left. - f(t_{i-1}', s_j') - f(t_i', s_{j-1}')|\right) + \epsilon.$$

Consequently, we get

$$V_\Phi^S(f^*, I_a^b) \le V_\Phi^S(2f, I_a^b) + \epsilon < \infty. \tag{9}$$

Finally, combining (7), (8) and (9) we derive

$$TV_\Phi^S(f^*) \le TV_\Phi^S(2f) < \infty,$$

which means that $f^* \in BV_{\Phi,*}^S(I_a^b)$.
Thus the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3   Main result

In this section we prove the main theorem of the paper.
This result characterizes the composition operator acting from the space $BV_\Phi^S(I_a^b)$ into itself which is Lipschitzian.

Theorem 3.1. *Let Φ be convex and let $H : BV_\Phi^S(I_a^b) \to BV_\Phi^S(I_a^b)$ be a composition operator generated by the function $h : I_a^b \times \mathbb{R} \to \mathbb{R}$, i.e.*

$$(Hf)(t,s) = h(t, s, f(t,s)), \ f \in \mathbb{R}^{I_a^b} \text{ for } (t,s) \in I_a^b.$$

If the operator H acts from the space $BV_\Phi^S(I_a^b)$ and is Lipschitzian, then

$$|h(x, u_1) - h(x, u_2)| \leq \delta|u_1 - u_2| \tag{10}$$

for all $x \in I_a^b$ and $u_1, u_2 \in \mathbb{R}$, where $\delta > 0$ is a constant.
Moreover, there exist functions $h_0, h_1 \in BV_{\Phi,*}^S(I_a^b)$ such that

$$h^*(x, u) = h_0(x) + h_1(x)u, \tag{11}$$

for $x \in I_a^b$ and $u \in \mathbb{R}$. Conversely, if $h_0, h_1 \in BV_{\Phi,*}^S(I_a^b)$ are functions such that (11) holds, then H acts from the space $BV_\Phi^S(I_a^b)$ into itself and is Lipschitzian.

Proof. Let us fix arbitrarily $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and define an auxiliary function $\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{for } t \leq \alpha, \\ \dfrac{t - \alpha}{\beta - \alpha} & \text{for } \alpha \leq t \leq \beta, \\ 1 & \text{for } t \geq \beta. \end{cases} \tag{12}$$

Keeping in mind that the operator $H : BV_\Phi^S(I_a^b) \to BV_\Phi^S(I_a^b)$ is Lipschitzian, we infer that there exists a constant $\mu > 0$ such that $\|Hf_1 - Hf_2\|_\Phi^S \leq \mu\|f_1 - f_2\|_\Phi^S$ for any $f_1, f_2 \in BV_\Phi^S(I_a^b)$. The definition of the norm implies

$$P_\Phi(Hf_1 - Hf_2) \leq \|Hf_1 - Hf_2\|_\Phi^S \leq \mu\|f_1 - f_2\|_\Phi^S. \tag{13}$$

In order to simplify the notation let us put $\mathcal{H} = Hf_1 - Hf_2$. Then, in view of (13) we get

$$P_\Phi(\mathcal{H}) \leq \|\mathcal{H}\|_\Phi^S \leq \mu\|f_1 - f_2\|_\Phi^S. \tag{14}$$

If $\|f_1 - f_2\|_\Phi^S > 0$ then from Lemma 2.1 (c) in (14) we have

$$TV_\Phi^S\left(\frac{\mathcal{H}}{\mu\|f_1 - f_2\|_\Phi^S}\right) \leq 1.$$

From the definition of TV_Φ^S, we infer that

$$\begin{aligned}
\phi_{n,m}\left(\left|\frac{\mathcal{H}(\cdot, c_1)}{\mu\|f_1 - f_2\|_\Phi^S}\right|\right) &\leq V_{\Phi,I_1}^S\left(\frac{\mathcal{H}(\cdot, x_2)}{\mu\|f_1 - f_2\|_\Phi^S}\right) \leq 1, \\
\phi_{n,m}\left(\left|\frac{\mathcal{H}(a_1, \cdot)}{\mu\|f_1 - f_2\|_\Phi^S}\right|\right) &\leq V_{\Phi,J_1}^S\left(\frac{\mathcal{H}(x_1, \cdot)}{\mu\|f_1 - f_2\|_\Phi^S}\right) \leq 1, \\
\phi_{n,m}\left(\left|\frac{\mathcal{H}(\cdot, \cdot)}{\mu\|f_1 - f_2\|_\Phi^S}\right|\right) &\leq V_{\Phi,I_a^b}^S\left(\frac{\mathcal{H}(\cdot, \cdot)}{\mu\|f_1 - f_2\|_\Phi^S}\right) \leq 1.
\end{aligned} \tag{15}$$

Thus, for any $u_1, u_2 \in \mathbb{R}$ and $a = (a_1, c_1)$, $b = (b_1, d_1)$, $x = (x_1, x_2) \in I_a^b$ we deduce

$$\begin{aligned}
|h(x_1, x_2, u_1) - h(x_1, x_2, u_2)| &= |\mathcal{H}(x)| \\
&= |\mathcal{H}(x_1, c_1) - \mathcal{H}(a_1, c_1) + \mathcal{H}(a_1, x_2) - \mathcal{H}(a_1, c_1) \\
&\quad + \mathcal{H}(a_1, c_1) - \mathcal{H}(a_1, x_2) - \mathcal{H}(x_1, c_1) + \mathcal{H}(x) + \mathcal{H}(a_1, c_1)| \\
&\leq |\mathcal{H}(x_1, c_1) - \mathcal{H}(a_1, c_1)| + |\mathcal{H}(a_1, x_2) - \mathcal{H}(a_1, c_1)| \\
&\quad + |\mathcal{H}(a_1, c_1) - \mathcal{H}(a_1, x_2) - \mathcal{H}(x_1, c_1) + \mathcal{H}(x)| + |\mathcal{H}(a_1, c_1)| \\
&\leq 3\phi_{n,m}^{-1}(1)\mu\|f_1 - f_2\|_\Phi^S + |\mathcal{H}(a_1, c_1)|. \tag{16}
\end{aligned}$$

To prove the inequality (10) we consider the following cases:

$$(\text{i}) \quad a_1 < x_1 \leq b_1 \quad \text{and} \quad c_1 < x_2 \leq d_1,$$
$$(\text{ii}) \quad a_1 < x_1 \leq b_1 \quad \text{and} \quad x_2 = c_1,$$
$$(\text{iii}) \quad x_1 = a_1 \quad \text{and} \quad c_1 < x_2 \leq d_1,$$
$$(\text{iv}) \quad x_1 = a_1 \quad \text{and} \quad x_2 = c_1.$$

Case (i). Consider the functions $f_1, f_2 \in BV_\Phi^S\left(I_a^b\right)$ defined by the formulas $f_\ell(y_1, y_2) := \left(\eta_{a_1, x_1}(y_1) + \eta_{c_1, x_2}(y_2)\right) u_\ell$ such that $a_1 \leq y_1 \leq b_1$, $c_1 \leq y_2 \leq d_1$ for $\ell = 1, 2$. Note that $f_\ell(a) = f_\ell(a_1, c_1) = 0$ and $f_\ell(x_1, c_1) = u_\ell$ for $\ell = 1, 2$, and $|\mathcal{H}(a)| = |\mathcal{H}(a_1, c_1)| = 0$.

Let $\varepsilon > 0$ such that $TV_\Phi^S\left(\frac{f_1 - f_2}{\varepsilon}\right) = 1$. Next, we get

$$
\begin{aligned}
V_{\Phi, I_1}^S\left(\frac{(f_1 - f_2)}{\varepsilon}(\cdot, c_1)\right) &= \sup\left\{\sum_{n=1}^\infty \phi_{n,m}\left(\left|\frac{(f_1 - f_2)}{\varepsilon}(I_n, c_1)\right|\right) : \{I_n\}\right\} \\
&= \sup\left\{\sum_{n=1}^\infty \phi_{n,m}\left(\left|\frac{(f_1 - f_2)}{\varepsilon}(b_n, c_1) - \frac{(f_1 - f_2)}{\varepsilon}(a_n, c_1)\right|\right) : \{I_n\}\right\} \\
&= \sup\left\{\sum_{n=1}^\infty \phi_{n,m}\left(\left|\frac{(b_n - a_n)}{\varepsilon(x_1 - a_1)}(u_1 - u_2)\right|\right) : \{I_n\}\right\} \\
&= \phi_{n,m}\left(\left|\frac{(b_1 - a_1)}{\varepsilon(x_1 - a_1)}(u_1 - u_2)\right|\right),
\end{aligned}
$$

$$
\begin{aligned}
&V_{\Phi, J_1}^S\left(\frac{(f_1 - f_2)}{\varepsilon}(a_1, \cdot)\right) \\
&\qquad = \sup\left\{\sum_{m=1}^\infty \phi_{n,m}\left(\left|\frac{(f_1 - f_2)}{\varepsilon}(a_1, J_m)\right|\right) : \{J_m\}\right\} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&V_{\Phi, I_a^b}^S\left(\frac{(f_1 - f_2)}{\varepsilon}(\cdot, \cdot)\right) \\
&\qquad = \sup\left\{\sum_{n=1}^\infty \sum_{m=1}^\infty \phi_{n,m}\left(\left|\frac{(f_1 - f_2)}{\varepsilon}(I_n, J_m)\right|\right) : \{I_n\}, \{J_m\}\right\} = 0.
\end{aligned}
$$

Hence
$$
1 = TV_\Phi^S\left(\frac{f_1 - f_2}{\varepsilon}\right) = \phi_{n,m}\left(\left|\frac{(b_1 - a_1)}{\varepsilon(x_1 - a_1)}(u_1 - u_2)\right|\right).
$$

Moreover, we deduce
$$
\varepsilon = \frac{|b_1 - a_1|\,|u_1 - u_2|}{\phi_{n,m}^{-1}(1)\,|x_1 - a_1|}.
$$

By virtue of Lemma 2.1 (d), we choose $P_\Phi(f_1 - f_2) = \varepsilon$ and derive

$$
\begin{aligned}
\|f_1 - f_2\|_\Phi^S &= |(f_1 - f_2)(a)| + P_\Phi(f_1 - f_2) \\
&= 0 + \frac{|b_1 - a_1| \, |u_1 - u_2|}{\phi_{n,m}^{-1}(1) \, |x_1 - a_1|}.
\end{aligned}
\tag{17}
$$

Now, employing (17) in the inequality (16) we obtain (10), i.e.

$$
\begin{aligned}
|h(x, u_1) - h(x, u_2)| &\leq 3\phi_{n,m}^{-1}(1)\mu \frac{|b_1 - a_1| \, |u_1 - u_2|}{\phi_{n,m}^{-1}(1) \, |x_1 - a_1|} + |\mathcal{H}(a_1, c_1)| \\
&= 3\mu \frac{|b_1 - a_1| \, |u_1 - u_2|}{|x_1 - a_1|} + 0 \\
&= \delta \, |u_1 - u_2| \qquad \left(\text{where } \delta = 3\mu \frac{|b_1 - a_1|}{|x_1 - a_1|} \right).
\end{aligned}
$$

Case (ii). Define the functions

$$
f_\ell(y_1, y_2) = (\eta_{a_1, x_1}(y_1)) \, u_\ell \text{ for } \ell = 1, 2, \ a_1 \leq y_1 \leq b_1, \ c_1 \leq y_2 \leq d_1.
\tag{18}
$$

Observe that $f_\ell(a) = f_\ell(a_1, c_1) = (\eta_{a_1, x_1}(a_1)) \, u_\ell = 0$ for $\ell = 1, 2$. As in the case (i), we get

$$
\begin{aligned}
V_{\Phi, I_1}^S \left(\frac{(f_1 - f_2)}{\varepsilon} (\cdot, c_1) \right) &= \sup \left\{ \sum_{n=1}^\infty \phi_{n,m} \left(\left| \frac{(f_1 - f_2)}{\varepsilon} (I_n, c_1) \right| \right) : \{I_n\} \right\} \\
&= \phi_{n,m} \left(\left| \frac{(b_1 - a_1)}{\varepsilon(x_1 - a_1)} (u_1 - u_2) \right| \right), \\
V_{\Phi, J_1}^S \left(\frac{(f_1 - f_2)}{\varepsilon} (a_1, \cdot) \right) &= 0 = V_{\Phi, I_a^b}^S \left(\frac{(f_1 - f_2)}{\varepsilon} (\cdot, \cdot) \right).
\end{aligned}
$$

Fix some arbitrary $\varepsilon > 0$ such that

$$
1 = TV_\Phi^S \left(\frac{f_1 - f_2}{\varepsilon} \right) = \phi_{n,m} \left(\left| \frac{(b_1 - a_1)}{\varepsilon(x_1 - a_1)} (u_1 - u_2) \right| \right).
$$

We obtain

$$
\varepsilon = \frac{|b_1 - a_1| \, |u_1 - u_2|}{\phi_{n,m}^{-1}(1) \, |x_1 - a_1|}.
$$

Taking $P_\Phi(f_1 - f_2) = \varepsilon$ and using the Lemma 2.1 (d) we get

$$
\|f_1 - f_2\|_\Phi^S = 0 + \frac{|b_1 - a_1| \, |u_1 - u_2|}{\phi_{n,m}^{-1}(1) \, |x_1 - a_1|}.
\tag{19}
$$

Employing (19) in the inequality (16) we get (10), i.e.

$$
\begin{aligned}
|h(x, u_1) - h(x, u_2)| &\leq 3\phi_{n,m}^{-1}(1)\mu \frac{|b_1 - a_1| \, |u_1 - u_2|}{\phi_{n,m}^{-1}(1) \, |x_1 - a_1|} + |\mathcal{H}(a_1, c_1)| \\
&= \delta \, |u_1 - u_2| \qquad \left(\text{where } \delta = 3\mu \frac{|b_1 - a_1|}{|x_1 - a_1|} \right).
\end{aligned}
$$

Case (iii). In this case we proceed in the analogous manner as in the case (ii), for which the functions $f_1, f_2 \in BV_\Phi^S \left(I_a^b \right)$ are defined by the formulas

$$f_\ell(y_1, y_2) = (\eta_{c_1, x_2}(y_2)) u_\ell \quad \text{for} \quad \ell = 1, 2, \ a_1 \le y_1 \le b_1, \ c_1 \le y_2 \le d_1.$$

Case (iv). Consider the functions $f_1, f_2 \in BV_\Phi^S \left(I_a^b \right)$ defined by

$$f_\ell(y_1, y_2) = [2 - \eta_{a_1, b_1}(y_1) - \eta_{c_1, d_1}(y_2)] u_\ell \quad \text{for} \quad \ell = 1, 2$$

such that $a_1 \le y_1 \le b_1$ and $c_1 \le y_2 \le d_1$.

Observe that

$$
\begin{aligned}
f_1(a) &= f_1(a_1, c_1) = [2 - \eta_{a_1, b_1}(a_1) - \eta_{c_1, d_1}(c_1)] u_1 = [2 - 0 - 0] u_1 = 2u_1, \\
f_2(a) &= f_2(a_1, c_1) = [2 - \eta_{a_1, b_1}(a_1) - \eta_{c_1, d_1}(c_1)] u_2 = [2 - 0 - 0] u_2 = 2u_2, \\
f_1(b) &= f_1(b_1, d_1) = [2 - \eta_{a_1, b_1}(b_1) - \eta_{c_1, d_1}(d_1)] u_1 = [2 - 1 - 1] u_1 = 0, \\
f_2(b) &= f_2(b_1, d_1) = [2 - \eta_{a_1, b_1}(b_1) - \eta_{c_1, d_1}(d_1)] u_2 = [2 - 1 - 1] u_2 = 0, \\
\mathcal{H}(a) &= h(a, u_1) - h(a, u_2), \\
\mathcal{H}(b) &= \mathcal{H}(b_1, d_1) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(f_1 - f_2)(a_1, J_m) &= [2 - \eta_{c_1, d_1}(d_m) - 2 + \eta_{c_1, d_1}(c_m)] (u_1 - u_2) \\
&= \left[-\frac{d_m - c_1}{d_1 - c_1} + \frac{c_m - c_1}{d_1 - c_1} \right] (u_1 - u_2) \\
&= \left[-\frac{d_m - c_m}{d_1 - c_1} \right] (u_1 - u_2), \\
(f_1 - f_2)(I_n, c_1) &= \left[-\frac{b_n - a_n}{b_1 - a_1} \right] (u_1 - u_2), \\
(f_1 - f_2)(I_n, J_m) &= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
V_{\Phi, I_1}^S \left(\frac{(f_1 - f_2)}{\varepsilon}(I_n, c_1) \right) &= \phi_{n,m} \left(\left| \frac{(u_1 - u_2)}{\varepsilon} \right| \right), \\
V_{\Phi, J_1}^S \left(\frac{(f_1 - f_2)}{\varepsilon}(a_1, J_m) \right) &= \phi_{n,m} \left(\left| \frac{(u_1 - u_2)}{\varepsilon} \right| \right), \\
V_{\Phi, I_a^b}^S \left(\frac{(f_1 - f_2)}{\varepsilon}(I_n, J_m) \right) &= 0.
\end{aligned}
$$

Therefore

$$TV_\Phi^S \left(\frac{f_1 - f_2}{\varepsilon} \right) = 2\phi_{n,m} \left(\left| \frac{(u_1 - u_2)}{\varepsilon} \right| \right).$$

Taking $\varepsilon > 0$ such that

$$1 = TV_\Phi^S \left(\frac{f_1 - f_2}{\varepsilon} \right) = 2\phi_{n,m} \left(\left| \frac{(u_1 - u_2)}{\varepsilon} \right| \right),$$

we get the following

$$\varepsilon = \frac{|u_1 - u_2|}{\phi_{n,m}^{-1}(1/2)}.$$

Now, we select $P_\Phi(f_1 - f_2) = \varepsilon$ and by virtue of the Lemma 2.1 (d) we get the result

$$\|f_1 - f_2\|_\Phi^S = \frac{2\phi_{n,m}^{-1}(1/2) + 1}{\phi_{n,m}^{-1}(1/2)}|u_1 - u_2|. \tag{20}$$

In consequence

$$
\begin{aligned}
|h(a_1, c_1, u_1) - h(a_1, c_1, u_2)| &= |\mathcal{H}(a)| \\
&= |\mathcal{H}(b_1, c_1) - \mathcal{H}(a_1, c_1) + \mathcal{H}(a_1, d_1) - \mathcal{H}(a_1, c_1) \\
&\quad + \mathcal{H}(a_1, c_1) - \mathcal{H}(a_1, d_1) - \mathcal{H}(b_1, c_1) + \mathcal{H}(b_1, d_1) - \mathcal{H}(b_1, d_1)| \\
&\leq |\mathcal{H}(b_1, c_1) - \mathcal{H}(a_1, c_1)| + |\mathcal{H}(a_1, d_1) - \mathcal{H}(a_1, c_1)| \\
&\quad + |\mathcal{H}(a_1, c_1) - \mathcal{H}(a_1, d_1) - \mathcal{H}(b_1, c_1) + \mathcal{H}(b_1, d_1)| + |\mathcal{H}(b_1, d_1)| \\
&\leq 3\phi_{n,m}^{-1}(1)\mu\|f_1 - f_2\|_\Phi^S + |\mathcal{H}(b_1, d_1)| \\
&= 3\phi_{n,m}^{-1}(1)\mu\frac{2\phi_{n,m}^{-1}(1/2) + 1}{\phi_{n,m}^{-1}(1/2)}|u_1 - u_2| + |0| \\
&= \delta|u_1 - u_2| \qquad \left(\text{where } \delta = 3\phi_{n,m}^{-1}(1)\mu\frac{2\phi_{n,m}^{-1}(1/2) + 1}{\phi_{n,m}^{-1}(1/2)}\right). \tag{21}
\end{aligned}
$$

From the foregoing cases we conclude that h is Lipschitzian.

Next, we show the estimation expressed in (11). Let us fix arbitrarily $x_1 \in (a_1, b_1]$, $x_2 \in (c_1, d_1]$ and put $x = (x_1, x_2) \in I_a^b$. For each $k \in \mathbb{N}$ we consider

$$a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \cdots < \alpha_k < \beta_k < x_1,$$

$$c_1 < \overline{\alpha_1} < \overline{\beta_1} < \overline{\alpha_2} < \overline{\beta_2} < \overline{\alpha_3} < \overline{\beta_3} < \cdots < \overline{\alpha_k} < \overline{\beta_k} < x_2$$

with $\eta_k : [a_1, b_1] \to [0, 1]$ and $\overline{\eta}_k : [c_1, d_1] \to [0, 1]$ two auxiliaries functions defined by the following formulas

$$\eta_k(t) := \begin{cases} 0 & \text{for } a_1 \leq t \leq \alpha_1, \\ \eta_{\alpha_i, \beta_i}(t) & \text{for } \alpha_i \leq t \leq \beta_i, \quad i = 1, 2, \ldots, k, \\ 1 - \eta_{\beta_i, \alpha_{i+1}}(t) & \text{for } \beta_i \leq t \leq \alpha_{i+1}, \quad i = 1, 2, \ldots, k-1, \\ 1 & \text{for } \beta_k \leq t \leq b_1 \end{cases} \tag{22}$$

and

$$\overline{\eta}_k(s) := \begin{cases} 0 & \text{for } c_1 \leq s \leq \overline{\alpha}_1, \\ \eta_{\overline{\alpha}_i, \overline{\beta}_i}(s) & \text{for } \overline{\alpha}_i \leq s \leq \overline{\beta}_i, \quad i = 1, 2, \ldots, k, \\ 1 - \eta_{\overline{\beta}_i, \overline{\alpha}_{i+1}}(s) & \text{for } \overline{\beta}_i \leq s \leq \overline{\alpha}_{i+1}, \quad i = 1, 2, \ldots, k-1, \\ 1 & \text{for } \overline{\beta}_k \leq s \leq d_1. \end{cases} \tag{23}$$

For any $u_1, u_2 \in \mathbb{R}$ we define the functions f_1, f_2 by

$$f_\ell(y_1, y_2) = [-\eta_k(y_1) + \overline{\eta}_k(y_2)] u_1 + (2 - \ell)u_2, \quad a_1 \le y_1 \le b_1, \quad c_1 \le y_2 \le d_1,$$

where $\ell = 1, 2$.

We denote the intervals I_{α_k} and $I_{\overline{\alpha}_k}$ by $I_{\alpha_k} = [\alpha_k, \beta_k] \subset [a_1, b_1] = I_1$ and $I_{\overline{\alpha}_k} = [\overline{\alpha}_k, \overline{\beta}_k] \subset [c_1, d_1] = J_1$, then

$$f_1(\cdot, \cdot) - f_2(\cdot, \cdot) = u_2 \quad \text{and} \quad \|f_1 - f_2\|_\Phi^S = |u_2|.$$

Using inequality (15) we have

$$\sum_{i=1}^{k} \phi_{i,m} \left(\left| \frac{H(\beta_i, \overline{\beta}_i) - H(\alpha_i, \overline{\beta}_i)}{\mu \|f_1 - f_2\|_\Phi^S} \right| \right)$$

$$\le \quad \sup \left\{ \sum_{k=1}^{\infty} \phi_{k,m} \left(\left| \frac{\mathcal{H}(I_{\alpha_k}, \overline{\beta}_k)}{\mu \|f_1 - f_2\|_\Phi^S} \right| \right) : \{I_{\alpha_k}\} \right\}$$

$$\le \quad V_{\Phi, I_1}^S \left(\frac{\mathcal{H}(I_{\alpha_k}, \overline{\beta}_k)}{\mu \|f_1 - f_2\|_\Phi^S} \right) \le 1.$$

Thus

$$\sum_{i=1}^{k} \phi_{i,m} \left(\frac{\left| h\left(\beta_i, \overline{\beta}_i, f_1(\beta_i, \overline{\beta}_i)\right) - h\left(\beta_i, \overline{\beta}_i, f_2(\beta_i, \overline{\beta}_i)\right) - h\left(\alpha_i, \overline{\beta}_i, f_1(\alpha_i, \overline{\beta}_i)\right)}{\mu |u_2|} \right.$$

$$\left. + \frac{h\left(\alpha_i, \overline{\beta}_i, f_2(\alpha_i, \overline{\beta}_i)\right)|}{\mu |u_2|} \right) \le 1.$$

Since $f_1(\beta_i, \overline{\beta}_i) = u_2$, $f_2(\beta_i, \overline{\beta}_i) = 0$, $f_1(\alpha_i, \overline{\beta}_i) = u_1 + u_2$, $f_2(\alpha_i, \overline{\beta}_i) = u_1$, we get from the foregoing estimation

$$\sum_{i=1}^{k} \phi_{i,m} \left(\frac{| h\left(\beta_i, \overline{\beta}_i, u_2\right) - h\left(\beta_i, \overline{\beta}_i, 0\right) - h\left(\alpha_i, \overline{\beta}_i, u_1 + u_2\right)}{\mu |u_2|} \right.$$

$$\left. + \frac{h\left(\alpha_i, \overline{\beta}_i, u_1\right)|}{\mu |u_2|} \right) \le 1. \qquad (24)$$

It is great importance to remark that the constant functions of two variables defined on the rectangle I_a^b belong to the space $BV_\Phi^S\left(I_a^b\right)$ since the composition operator H generated by h acts from $BV_\Phi^S\left(I_a^b\right)$ into $BV_\Phi^S\left(I_a^b\right)$ and the functions $h(\cdot, u)[x \mapsto h(x, u)]$ belong to the space $BV_\Phi^S\left(I_a^b\right)$ for each $u \in \mathbb{R}$. On the other hand, we know from Lemma 2.2 that the regularization left-left in the first two variables $h^*(\cdot, u)$ belongs to the space $BV_{\Phi,*}^S\left(I_a^b\right)$ for all $u \in \mathbb{R}$. If we apply limit in (24) when $(\alpha_1, \overline{\alpha}_1) \mapsto (x_1 - 0, x_2 - 0)$ we obtain

$$\sum_{i=1}^{k} \phi_{i,m} \left(\frac{| h^*\left(x_1, x_2, u_2\right) - h^*\left(x_1, x_2, 0\right) - h^*\left(x_1, x_2, u_1 + u_2\right)}{\mu |u_2|} \right.$$

$$\left. + \frac{h^*\left(x_1, x_2, u_1\right)|}{\mu |u_2|} \right) \le 1.$$

Without losing generality we fix $i = n$ for $n = 1, 2, \ldots, k$

$$k\phi_{n,m}\left(\frac{|\,h^*\,(x_1, x_2, u_2) - h^*\,(x_1, x_2, 0) - h^*\,(x_1, x_2, u_1 + u_2)}{\mu|u_2|}\right.$$
$$\left. + \frac{h^*\,(x_1, x_2, u_1)\,|}{\mu|u_2|}\right) \leq 1.$$

Hence

$$\phi_{n,m}\left(\frac{|\,h^*\,(x_1, x_2, u_2) - h^*\,(x_1, x_2, 0) - h^*\,(x_1, x_2, u_1 + u_2)}{\mu|u_2|}\right.$$
$$\left. + \frac{h^*\,(x_1, x_2, u_1)\,|}{\mu|u_2|}\right) \leq \frac{1}{k}.$$

Since $k \in \mathbb{N}$ is arbitrary we derive

$$\phi_{n,m}\left(\frac{|\,h^*\,(x_1, x_2, u_2) - h^*\,(x_1, x_2, 0) - h^*\,(x_1, x_2, u_1 + u_2)}{\mu|u_2|}\right.$$
$$\left. + \frac{h^*\,(x_1, x_2, u_1)\,|}{\mu|u_2|}\right) = 0.$$

Because $\phi_{n,m}$ is convex for $n, m = 1, 2, \ldots$ and $\phi(t) = 0$ only if $t = 0$, then

$$\frac{|\,h^*\,(x_1, x_2, u_2) - h^*\,(x_1, x_2, 0) - h^*\,(x_1, x_2, u_1 + u_2) + h^*\,(x_1, x_2, u_1)\,|}{\mu|u_2|} = 0.$$

Therefore

$$h^*\,(x_1, x_2, u_2) - h^*\,(x_1, x_2, 0) - h^*\,(x_1, x_2, u_1 + u_2) + h^*\,(x_1, x_2, u_1) = 0$$

or equivalently

$$h^*\,(x, u_1 + u_2) + h^*\,(x, 0) = h^*\,(x, u_1) + h^*\,(x, u_2) \tag{25}$$

for each $x = (x_1, x_2) \in (a_1, b_1] \times (c_1, d_1]$ and all $u_1, u_2 \in \mathbb{R}$.
Let $x_1 \in (a_1, b_1]$ and $x_2 = c_1$, now we consider the following inequalities

$$a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \cdots < \alpha_k < \beta_k < x_1,$$

$$c_1 < \overline{\alpha_1} < \overline{\beta_1} < \overline{\alpha_2} < \overline{\beta_2} < \overline{\alpha_3} < \overline{\beta_3} < \cdots < \overline{\alpha_k} < \overline{\beta_k} < d_1, \ k \in \mathbb{N}.$$

We proceed in the similar way as in the result (24). Taking limit when $(\alpha_1, \overline{\beta}_1) \mapsto (x_1 - 0, x_2 + 0)$ in (24) we obtain (25). The cases $x_1 = a_1$ and $x_2 \in (c_1, d_1]$ or $x_1 = a_1$ and $x_2 = c_1$ are similar.
Thus the equation (25) holds for each $x = (x_1, x_2) \in I_a^b$ and for any $u_1, u_2 \in \mathbb{R}$.
Now, we fix $x = (x_1, x_2) \in I_a^b$ and define the mapping $T_x : \mathbb{R} \to \mathbb{R}$ by

$$T_x(u) = h^*(x, u) - h^*(x, 0) \quad \forall u \in \mathbb{R}.$$

Note that we can rewrite the expression in (25) as follows

$$T_x(u_1 + u_2) = T_x(u_1) + T_x(u_2) \quad \forall u_1, u_2 \in \mathbb{R}. \tag{26}$$

This shows that T_x is an additive operator.
For any $u_1, u_2 \in \mathbb{R}$ we have

$$
\begin{aligned}
|T_x(u_1) - T_x(u_2)| &= |h^*(x, u_1) - h^*(x, 0) - h^*(x, u_2) + h^*(x, 0)| \\
&= |h^*(x, u_1) - h^*(x, u_2)| \\
&\leq \mu |u_1 - u_2|,
\end{aligned}
$$

i.e. $T_x(\cdot)$ is Lipschitz-continuous on $\mathbb{R}$. Then exists a mapping $h_1 : I_a^b \to \mathbb{R}$ such that

$$T_x(u) = h_1(x)u \quad \forall x \in I_a^b, \quad \forall u \in \mathbb{R}.$$

Taking $h_0(x) = h^*(x, 0)$, $x \in I_a^b$ we derive

$$h^*(x, u) = T_x(u) + h^*(x, 0) = h_1(x)u + h_0(x).$$

Since $h_0(\cdot) = h^*(\cdot, 0)$, $h_1(\cdot) = h^*(\cdot, 1) - h^*(\cdot, 0)$ and Lemma 2.2 we have that $h_0, h_1 \in BV_{\Phi,*}^S(I_a^b)$. Thus

$$h^*(x, u) = h_1(x)u + h_0(x) \quad \forall x \in I_a^b, \forall u \in \mathbb{R} \text{ with } h_0, h_1 \in BV_{\Phi,*}^S(I_a^b).$$

Sufficient Condition. Suppose that the composition operator H is given by

$$(Hf)(x) = h_0(x) + h_1(x)f(x), \quad x \in I_a^b, \quad f \in BV_\Phi^S(I_a^b).$$

As $BV_\Phi^S(I_a^b)$ is a Banach algebra, then H maps the space $BV_\Phi^S(I_a^b)$ into itself. Further

$$
\begin{aligned}
\|H(f_1) - H(f_2)\|_\Phi^S &= \|h_0 + h_1 f_1 - h_0 - h_1 f_2\|_\Phi^S \\
&\leq K \|h_1\|_\Phi^S \|f_1 - f_2\|_\Phi^S \\
&= \lambda \|f_1 - f_2\|_\Phi^S \quad \left(\text{where } \lambda = K \|h_1\|_\Phi^S\right). \tag{27}
\end{aligned}
$$

In consequence H is a Lipschitzian operator. □
Remark 3.1.

1) The Theorem 3.1 is valid for the regularization right-right, left-right and right-left of $h(\cdot, u)$ $\forall u \in \mathbb{R}$.

2) If $h_0, h_1 \in BV_{\Phi,*}^S(I_a^b)$ and $\|h_1\|_\Phi^S < 1/K$, then by Principium of contraction of Banach in combination with (27), exists only one function $f \in BV_\Phi^S(I_a^b)$ such that

$$f(x) = h_0(x) + h_1(x)f(x) \quad \forall x \in I_a^b \subset \mathbb{R}.$$

The following corollary is the immediate consequence of the Theorem 3.1.

Corollary 3.1. *Suppose that $h : I_a^b \times \mathbb{R} \to \mathbb{R}$ is such that $h^* = h$ in $I_a^b \times \mathbb{R} \to \mathbb{R}$, and composition operator H maps space $BV_\Phi^S \left(I_a^b \right)$ into itself. Then it is Lipschitzian if and only if there exist functions $h_0, h_1 \in BV_{\Phi,*}^S \left(I_a^b \right)$ such that*

$$h(x,u) = h_0(x) + h_1(x)u \quad \forall x \in I_a^b, \quad u \in \mathbb{R}.$$

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Interior controllability of the Benjamin-Bona-Mahony equation [1]

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ABSTRACT: In this paper we prove the interior approximate controllability of the following Generalized Benjamin-Bona-Mahony type equation (BBM) with homogeneous Dirichlet boundary conditions

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x), & t \in (0,\tau), \quad x \in \Omega, \\ z(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $I\!\!R^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed control $u \in L^2(0,\tau; L^2(\Omega))$. We prove that for all $\tau > 0$ and any nonempty open subset ω of Ω the system is approximately controllable on $[0,\tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time. As a consequence of this result we obtain the interior approximate controllability of the heat equation by putting $a = 0$ and $b = 1$.

1 Introduction.

The original Benjamin-Bona-Mahony equation was proposed in [4] for the case $N = 1$ as a model for the propagation of long waves. This equation and related types of pseudo-parabolic equations have been studied by many authors. Results about existence and uniqueness of solutions can be found in [3]; the long time behavior of solutions and the existence of attractors were studied e.g. in [5], [7], [8] and [15], and

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the controllability for the case $N = 1$ with control in the boundary has been studied in [13]. Recently the BBM equation with boundary conditions has been studied in [6] and [12].

The interior approximate controllability is a well known, fascinating and important subject in systems theory; there are some works done by [14], [16], [17], [18] and [19]. Particularly, Zuazua in [19] proves the interior approximate controllability of the heat equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t,x), & \text{in} \quad (0,\tau) \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

in two different ways. In the first one, he uses the Hahn-Banach theorem, integration by parts, the adjoint equation, the Carleman estimates and the Holmgren Uniqueness Theorem([11]).

The second method is constructive and uses a variational technique: fix the control time $\tau > 0$, the initial and final state $z_0 = 0$, $z_1 \in L^2(\Omega)$ respectively and $\epsilon > 0$; the control steering the initial state z_0 to a ball of radius $\epsilon > 0$ and center z_1 is given by the point in which the following functional achieves its minimum value

$$J_\epsilon(\varphi_\tau) = \frac{1}{2} \int_0^\tau \int_\omega \varphi^2 dx dt + \epsilon \|\varphi_\tau\|_{L^2(\Omega)} - \int_\Omega z_1 \varphi_\tau,$$

where φ is the solution of the corresponding adjoint equation with initial data φ_τ.

In this paper we prove the interior approximate controllability of the following Generalized Benjamin-Bona-Mahony type equation (BBM) with homogeneous Dirichlet boundary conditions

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x), & t \in (0,\tau), \quad x \in \Omega, \\ z(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $I\!\!R^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed control $u \in L^2(0,\tau; L^2(\Omega))$.

The controllability of such systems, with the controls acting on the whole set Ω was studied in [1]; they considered the approximate controllability of the system

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = b_1(x)u_1 + \ldots + b_m(x)u_m, & t \geq 0, \quad x \in \Omega, \\ z(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $b_i \in L^2(\Omega; I\!\!R)$, the control functions $u_i \in L^2(0,\tau; I\!\!R)$; $i = 1,2,\ldots,m$ and Ω is a bounded domain in $I\!\!R^N(N \geq 1)$. More precisely, they prove the following result: the system (1.3) is approximately controllable on $[0,\tau]$, $\tau > 0$ iff each of the following finite dimensional systems are controllable on $[0,\tau]$

$$y' = -\frac{b\lambda_j}{1 + a\lambda_j}y + B_j u, \quad y \in R(E_j), \quad j = 1,2,\ldots,\infty, \tag{1.4}$$

where

$$B_j : I\!\!R^m \to R(E_j), \quad B_j U = \sum_{i=1}^{\gamma_j} \frac{1}{1 + a\lambda_j} E_j b_i U_i,$$

λ_j' s are the eigenvalues of $-\Delta$ with Dirichlet boundary condition and γ_j the corresponding multiplicity, E_j' s are the projections on the corresponding eigenspaces and $R(E_j)$ denotes the range of E_j . Since $dimR(E_j) = \gamma_j < \infty$, the controllability of (1.4) is equivalent to the following algebraic condition:

$$Rank[B_j] = \gamma_j, \quad j = 1, 2, \ldots, \infty. \tag{1.5}$$

In this paper, we are interested in the interior approximate controllability of system (1.2). This is an important problem from the applications point of view, and more general since the control is acting only on a subset ω of Ω. We prove that for all $\tau > 0$ and any nonempty open subset ω of Ω the system is approximately controllable on $[0, \tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 3.2). As a consequence of this result we obtain the interior approximate controllability of the heat equation (1.1) by putting $a = 0$ and $b = 1$.

The technique given here is simple and based on the following results:

Theorem 1.1 *(see Theorem 1.23 from [2], pg. 20) Suppose $\Omega \subset I\!\!R^n$ is an open, non-empty and connected set, and f is real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω. Then, $f = 0$ in Ω.*

Lemma 1.1 *(see Lemma 3.14 from [9], pg. 62) Let $\{\alpha_j\}_{j\geq1}$ and $\{\beta_{i,j} : i = 1, 2, \ldots, m\}_{j\geq1}$ be two sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, \tau], \quad i = 1, 2, \cdots, m$$

iff

$$\beta_{i,j} = 0, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, \infty.$$

Theorem 1.2 *The eigenfunctions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω.*

2   Abstract Formulation of the Problem

In this section we choose the space in which this problem will be set as an abstract ordinary differential equation.

Let $Z = L^2(\Omega) = L^2(\Omega, I\!\!R)$ and consider the linear unbounded operator $A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, I\!\!R) \cap H_0^1(\Omega, I\!\!R).$$

The operator A has the following very well known properties: the spectrum of A consists of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \qquad \text{with} \qquad \lambda_j \to \infty, \tag{2.1}$$

each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore:

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of A.

b) for all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_j z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.3}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z. \tag{2.4}$$

c) $-A$ generates the analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At} z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \tag{2.5}$$

Hence, the equation (1.3) can be written as an abstract ordinary differential equation in Z as follows

$$z' + aAz' + bAz = 1_\omega u(t), \quad t \in (0, \tau]. \tag{2.6}$$

Since $(I + aA) = a(A - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(A)(\rho(A)$ is the resolvent set of A), then the operator:

$$I + aA : D(A) \to Z$$

is invertible with bounded inverse

$$(I + aA)^{-1} : Z \to D(A).$$

Therefore, the equation (2.6) also can be written as follows

$$z' + b(I + aA)^{-1} Az = (I + aA)^{-1} 1_\omega u(t) \quad t \in (0, \tau). \tag{2.7}$$

Moreover, $(I + aA)$ and $(I + aA)^{-1}$ can be written in terms of the eigenvalues of A:

$$(I + aA)z = \sum_{j=1}^{\infty} (1 + a\lambda_j) E_j z$$

$$(I + aA)^{-1}z = \sum_{j=1}^{\infty} \frac{1}{1 + a\lambda_j} E_j z.$$

Therefore, if we put $B = (I + aA)^{-1}$, the equation (2.7) can be written as follows

$$z' + bBAz = BB_\omega u(t), \quad t \in (0, \tau), \tag{2.8}$$

where $B_\omega f = 1_\omega f$ is a linear a bounded operator from Z to Z and $u \in L^2(0, \tau; L^2(\Omega)) = L^2(0, \tau; Z)$.

Now, we formulate a simple proposition.

Proposition 2.1 *The operators bBA and $T(t) = e^{-bBAt}$ are given by the following expression*

$$bBAz = \sum_{j=1}^{\infty} \frac{b\lambda_j}{1 + a\lambda_j} E_j z \tag{2.9}$$

$$T(t)z = e^{-bBAt}z = \sum_{j=1}^{\infty} e^{\frac{-b\lambda_j}{1 + a\lambda_j}t} E_j z, \tag{2.10}$$

and

$$\| T(t) \| \le e^{-\beta t}, \quad t \ge 0, \tag{2.11}$$

where

$$\beta = \inf_{j \ge 1} \left\{ \frac{b\lambda_j}{1 + a\lambda_j} \right\} = \frac{b\lambda_1}{1 + a\lambda_1}. \tag{2.12}$$

With this notation the system (2.8) can be written as follows

$$z' = -\mathcal{A}z + BB_\omega u(t), \quad t \in (0, \tau], \tag{2.13}$$

where $\mathcal{A} = bBA$.

3   Main Theorem

In this section we shall prove the main result of this paper on the controllability of the linear system (2.13). But first we give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z$ and a control $u \in L^2(0, \tau; Z)$ the equation (2.13) with $z(0) = z_0$ has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t - s)BB_\omega u(s)ds, \quad 0 \le t \le \tau. \tag{3.1}$$

Definition 3.1 *We say that (2.13) is approximately controllable in $[0, \tau]$ if for all z_0, $z_1 \in Z$ and $\epsilon > 0$, there exists a control $u \in L^2(0, \tau; Z)$ such that the solution $z(t)$ given by (3.1) satisfies*

$$\|z(\tau) - z_1\| \le \epsilon. \tag{3.2}$$

Consider the following bounded linear operator:

$$G : L^2(0, \tau; Z) \to Z, \quad Gu = \int_0^\tau T(\tau - s)BB_\omega u(s)ds, \qquad (3.3)$$

whose adjoint operator $G^* : Z \longrightarrow L^2(0, \tau; Z)$ is given by

$$(G^* z)(s) = (BB_\omega)^* T^*(\tau - s)z = B_\omega^* B^* T^*(\tau - s)z, \quad \forall s \in [0, \tau], \quad \forall z \in Z. \qquad (3.4)$$

The following lemma is trivial:

Lema 3.1 *The equation (2.13) is approximately controllable on $[0, \tau]$ if, and only if, $\overline{\text{Rang}(G)} = Z$.*

The following result is well known from linear operator theory:

Lema 3.2 *Let W and Z be Hilbert spaces and $G^* \in L(Z, W)$ the adjoint operator of the linear operator $G \in L(W, Z)$. Then*

$$\overline{\text{Rang}(G)} = Z \iff Ker(G^*) = \{0\}.$$

As a consequence of the foregoing Lemma one can prove the following result:

Lema 3.3 *Let W and Z be Hilbert spaces and $G^* \in L(Z, W)$ the adjoint operator of the linear operator $G \in L(W, Z)$. Then $\overline{\text{Rang}(G)} = Z$ if, and only if, one of the following statements holds:*

a) $Ker(G^*) = \{0\}$.

b) $\langle GG^* z, z \rangle > 0$, $z \neq 0$ in Z.

c) $\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1} z = 0$.

d) $\sup_{\alpha > 0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1$.

The following theorem follows directly from (3.4), lemma 3.1 and lemma 3.3.

Theorem 3.1 *(2.13) is approximately controllable on $[0, \tau]$ iff*

$$B_\omega^* B^* T^*(t)z = 0, \quad \forall t \in [0, \tau], \quad \Rightarrow z = 0. \qquad (3.5)$$

Theorem 3.2 *(Main Result) For all $\tau > 0$ and any open nonempty subset ω of Ω the system (2.13) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (2.13) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B_\omega^* B^* T(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0),$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).$$

Proof. We shall apply Theorem 3.1 to prove the controllability of system (2.13). To this end, we observe that

$$T^*(t)z = \sum_{j=1}^{\infty} e^{\frac{-b\lambda_j}{1+a\lambda_j}t} E_j z, \quad B_\omega^* = B_\omega^* \quad \text{and} \quad B^* = B.$$

Then,

$$(BB_\omega)^* T^*(t)z = B_\omega B T^*(t)z = \sum_{j=1}^{\infty} e^{\frac{-b\lambda_j}{1+a\lambda_j}t} \frac{1}{1+a\lambda_j} B_\omega E_j z = 0, \quad \forall t \in [0,\tau].$$

Since $\{\frac{-b\lambda_j}{1+a\lambda_j} : j = 1, 2, \dots\}$ is a decreasing sequence, then from Lemma 1.1 we obtain that

$$(B_\omega E_j z)(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > 1_\omega \phi_{j,k}(x) = 0, \quad \forall x \in \Omega, \quad j = 1, 2, \dots.$$

i.e.,

$$\sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0 \quad \forall x \in \omega, \quad j = 1, 2, \dots..$$

Now, from theorem 1.2 we know that $\phi_{j,k}$' s are analytic functions, which implies the analyticity of $E_j z$. Then, from Theorem 1.1 we get that

$$\sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0 \quad \forall x \in \Omega, \quad j = 1, 2, \dots..$$

Hence, $E_j z = 0$, $j = 1, 2, \dots$, which implies that $z = 0$.
Now, given the initial and the final states z_0 and z_1, we consider the sequence of controls

$$
\begin{aligned}
u_\alpha(\cdot) &= B_\omega^* B^* T(\tau - \cdot)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\
&= G^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \quad \alpha > 0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
Gu_\alpha &= GG^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\
&= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\
&= z_1 - T(\tau)z_0 - \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).
\end{aligned}
$$

From part c) of Lemma 3.3 we know that

$$\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) = 0.$$

Therefore,

$$\lim_{\alpha \to 0^+} Gu_\alpha = z_1 - T(\tau)z_0.$$

i.e.,

$$\lim_{\alpha \to 0^+} \{T(\tau)z_0 + \int_0^\tau T(\tau - s)BB_\omega u(s)ds\} = z_1.$$

This completes the proof of the Theorem.

□

Corollary 3.1 *For all $\tau > 0$ and all open nonempty subset ω of Ω the heat equation (1.1) is approximately controllable on $[0, \tau]$.*

Proof. It is enough to take $a = 0$ and $b = 1$ in the equation (1.2).   □

4   Final Remarks

The original Benjamin -Bona-Mohany Equation is a non-linear one, here we have proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBM equation. So, our next work is concerned with the controllability of non linear BBM equation

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f(t,z,u(t)), & t \in (0,\tau), \qquad x \in \Omega, \\ z(t,x) = 0, \quad t \geq 0, \qquad x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $I\!\!R^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0,\tau; L^2(\Omega))$ and $f(t,z,u(t))$ is a nonlinear perturbation.

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The Demyanov metric
and measurable multifunctions[1]

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ABSTRACT: We present in this paper measurability multifunctions in the family of all convex, bounded sets which need not be closed. The Demyanov metric is discussed.

1 Introduction and Preliminares

We introduction the following family subsets of $\mathbb{R}^d$:

$$\mathcal{C}^d = \left\{ A \in \mathbb{R}^d \ : \ A \neq \emptyset, convex, bounded \right\} \quad \mathcal{K}^d = \left\{ A \in \mathcal{C}^d \ : \ A \ is \ compact \right\}$$

For any $A \in \mathbb{R}^d$, $u \in \mathbb{R}^d$ we denote

$$p_A(u) = \sup_{a \in A} < a, u > \quad , \quad A(u) = \{ a \in A : < a, u > = p_A(u) \}$$

where $< . >$ is the scalar product, and by recurrence

$$A(u_1, ..., u_i) = A(u_1, ..., u_{i-1})(u_i).$$

By $\mathcal{E}$ denote the set of all orthonormal sequences $(e_1, ..., e_k), 1 \leq k \leq d$. We shall often use a single letter to denote elements of $\mathcal{E}$, like $E = (e_1, ..., e_k)$.

Let $A, B \subset \mathbb{R}^d$. The Hausdorff distance is defined by

$$\rho_H(A, B) = \max \{ e(A, B), e(B, A) \}$$

---

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where

$$e(A, B) = \sup_{a \in A} \text{dist}(a, B) = \sup_{a \in A} \inf_{b \in B} ||a - b||$$

This is a metric in the family of closed sets in $\mathbb{R}^d$.

In the [4] we give the following formula for the Demyanov's metric which permitts further extensions to the case of convex but not necessarily closed sets. Let

$$\mathcal{E}^k = \{(e_1, ..., e_j) \in \mathcal{E} \ : \ j \geq k\}$$

For $A, B \in \mathcal{K}^d$ and $1 \leq k \leq d$

$$\rho_D(A, B) = \sup_{E \in \mathcal{E}^k} \rho_H(A(E), B(E))$$

The metric is that of uniform convergence in the set of mappings whose arguments are orthonormal systems of vectors in $\mathbb{R}^d$ and values are convex sets orthogonal to these vectors.

2 Demyanov's metric in $\mathcal{C}^d$ and $\mathcal{X}_{\mathcal{U}}$

Let $\mathcal{E}_0 = \mathcal{E} \cup \{0\}$. We introduce in $\mathcal{C}^d$ the following equivalence relation.

Definition 2.1 $A \equiv B$ *iff for every* $E \in \mathcal{E}_0$ *we have*

$$A(E) \neq \emptyset \Leftrightarrow B(E) \neq \emptyset$$

Remark that the set $\mathcal{U}$ of $E \in \mathcal{E}_0$ for which $A(E) \neq \emptyset$ is common for all elements of the same equivalence class-this equivalence class will be then denoted as $\mathcal{K}_{\mathcal{U}}^d$.

If a set $\mathcal{U} \subset \mathcal{E}_0$ corresponds to some equivalence class then it will be called admissible. Any admissible set $\mathcal{U}$ satisfies the following two conditions:

(i) $0 \in \mathcal{U}$

(ii) $(e_1, ..., e_k, e_{k+1}) \in \mathcal{U} \Rightarrow (e_1, ..., e_k) \in \mathcal{U}$

The following example showing that the conditions (i),(ii) are not sufficient for a set $\mathcal{U}$ to be admissible.

Example 2.1 *Let* $d = 2$ *and* $\mathcal{U} = \{0\} \cup S^1$

Let $\mathcal{K}_{\mathcal{U}}^d \neq \emptyset$. We remark that the closed ball of $\mathbb{R}^d$ is an element of $\mathcal{K}_{\mathcal{U}}^d$. Let $A \in \mathcal{K}_{\mathcal{U}}^d$ and e_1 be such that $\bar{A}(e_1)$ is an exposed point in $\bar{A}$-the bar over A denotes the closure. Then $\bar{A}(e_1) \in A$. For any $e_2 \in S^1$ orthogonal to e_1 we have $A(e_1) = A(e_1, e_2) \neq \emptyset$, hence $(e_1, e_2) \notin \mathcal{U}$.

Remark that the family off all $A(E), E \in \mathcal{E}_0$ consists of all faces of the set $A \in \mathcal{C}^d$. The following counterpart of Theorem 2.1.2 [5] is valid for arbitrary $A \in \mathcal{C}^d$ (the $relint A$ denote the relative interior of set A).

Theorem 2.1 *If $A(E) \neq A(F)$ then $relint A(E) \cap relint A(F) = \emptyset$. Morever, the family of sets $relint A(E)$, for all $E \in \mathcal{E}_0$ provides a decomposition of A.*

Fix an admissible family $\mathcal{U}$. Let $E = (e_0, e_1, ..., e_k) \in \mathcal{U}$, where $0 \leq k \leq d$. By $\mathcal{Z}_E$ we denote the family of all convex, nonempty, relatively open sets A such that

$$\forall e_i \in E, \forall u, v \in A : \; < u - v, e_i >= 0$$

By $\mathcal{A}_E$ denote the element of $\mathcal{Z}_E$. We prove the following lemma:

Lemma 2.1 *The space $(\mathcal{Z}_E, \rho_H)$ is a complete.*

Proof: Let $E = (e_0, e_1, ..., e_k) \in \mathcal{U}$ where $0 \leq k \leq d$. We consider the Cauchy sequence $(\mathcal{A}_E^n)$. Then also $(\bar{\mathcal{A}}_E^n)$ is a Cauchy sequence and is element of $\mathcal{K}^d$. The space $(\mathcal{K}^d, \rho_H)$ is complete. Thus $\lim_{n \to \infty} \rho_H(\bar{\mathcal{A}}_E^n, B) = 0$, where $B \in \mathcal{K}^d$.

If $\dim B < d - k$, then the limit is equal B. If $\dim B \geq d - k$ then the limit is equal relintB.

The set of all elements $\mathcal{A}$ of the Cartezian product $\prod_{E \in \mathcal{U}} \mathcal{Z}_E$ for which the union $\bigcup_{E \in \mathcal{U}} \mathcal{A}_E$ is a bounded subset of $\mathbb{R}^d$ will by denoted by $\mathcal{X}_\mathcal{U}$.

Example 2.2 *Let $d = 2$ and $\mathcal{U} = \{e_0, (e_o, e_1), (e_0, e_1, e_2)\}$, where $e_0 = \{0\}, e_1 = (0, 1), e_2 = (1, 0)$. We define $\mathcal{A} \in \mathcal{X}_\mathcal{U}$:*

$$\mathcal{A}_{e_0} = intco\{(1,1), (-1,1), (-1,-1), (1,-1)\}$$

$$\mathcal{A}_{(e_0,e_1)} = relintco\left\{(\frac{1}{2}, 1), (1, 1)\right\}$$

$$\mathcal{A}_{(e_0,e_2)} = relintco\left\{(1, -\frac{1}{2}), (1, \frac{1}{2})\right\}$$

$$\mathcal{A}_{(e_0,e_1,e_2)} = \{(1,1)\}$$

The union of all $\mathcal{A}_E$ is not convex. Remark that putting

$$\mathcal{A}_{(e_0,e_1)} = relintco\left\{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\right\}$$

and

$$\mathcal{A}_{(e_0,e_1,e_2)} = \left\{(\frac{1}{2}, 1)\right\}$$

we get $\mathcal{A}$ for which the union of values is a convex set belonging to $\mathcal{K}_\mathcal{U}$,

In $\mathcal{X}_\mathcal{U}$ we introduce the following metric

$$\rho_P(\mathcal{A}, \mathcal{B}) = \sup_{E \in \mathcal{U}} \rho_H(\mathcal{A}_E, \mathcal{B}_E)$$

The following lemma is a standard result about completeness of the space of bounded maps with the metric uniform convergence.

Lemma 2.2 *The metric space $(\mathcal{X}_\mathcal{U}, \rho_P)$ is complete*

Proof: Let $\mathcal{U}$ the admissible set and $(\mathcal{A}^n)$ the Cauchy sequence in $\mathcal{X}_{\mathcal{U}}$. Using the definition of the metric ρ_P we have that for all $E \in \mathcal{U}$, $(\mathcal{A}_E^n)$ is the Cauchy sequence in $\mathcal{Z}_E$. Hence for all $E \in \mathcal{U}$, $\lim_{n\to\infty} \rho_H(\mathcal{A}_E^n, \mathcal{A}_E^0) = 0$, where $\mathcal{A}_E^0 \in \mathcal{Z}_E$. Let $\mathcal{A}^0 = \bigcup_{E\in\mathcal{U}} \mathcal{A}_E^0$.

We have that $\mathcal{A}^0 \in \mathcal{Z}_{\mathcal{U}}$ and $\lim_{n\to\infty} \rho_P(\mathcal{A}^n, \mathcal{A}^0) = 0$

We remark the following fact. The theorem 2.1 says that for $A \in \mathcal{K}_{\mathcal{U}}^d$ the union $\bigcup_{E\in\mathcal{U}} \mathcal{A}_E$ is equal A, so it is convex.

3 Measurable multifunctions in $\mathcal{X}_{\mathcal{U}}$

By $(T, \mathcal{M}, \mu)$ will denote a measurable space, i.e., T is a set, $\mathcal{M}$ is a σ-field and μ is a measure such that $\mu(A) < \infty$.

Definition 3.1 $F : T \to \mathcal{X}_{\mathcal{U}}$ *is simple if F takes only finitely many values $A_1, ..., A_k$ such that*

$$\{t : F(t) = A_i\} \in \mathcal{M} \ \ for \ \ i = 1, ..., k$$

Definition 3.2 *A multifunction $F : T \to \mathcal{X}_{\mathcal{U}}$ is a measurable if there are simple multifunctions $F_n : T \to \mathcal{X}_{\mathcal{U}}$ such that*

$$\lim_{n\to\infty} \rho_P(F_n(t), F(t)) = 0 \ \ a.e. \ \ t \in T$$

We can now proof the following result

Theorem 3.1 *If $F : T \to \mathcal{K}^d$ is measurable, then for any $A \in \mathcal{K}^d$ the set*

$$T_A = \{t : F(t) \cap A \neq \emptyset\} \in \mathcal{M}$$

and the multifunction $F_A : T_A \to \mathcal{K}^d$ defined by $F_A(t) = F(t) \cap A$ is measurable

Proof: Let $A_n = \left\{ x \in \mathbb{R}^d : \ dist(x, A) \leq \frac{1}{n} \right\}$ for $n = 1, 2, ...$, and F_n be a sequence of simple multifunctions such that

$$\lim_{n\to\infty} \rho_P(F_n(t), F(t)) = 0 \ \ a.e.$$

We define

$$T_n = \bigcup_j \bigcap_{i=j} \{t : F_i(t) \cap A_n \neq \emptyset\}$$

Remark that if $t_0 \in T_A$ then $F(t_0) \cap A \neq \emptyset$, and for all n exist $i_0(n)$ such that $F_i(t_0) \cap A_n \neq \emptyset$ for $i \geq i_0(n)$. So $t_0 \in T_n$ for any n, hence $T_A \subset \bigcap_n T_n$.

We now prove that $\bigcap_n T_n \subset T_A$.

Let $t_0 \in \bigcap_n T_n$, then for any n exist $j_0(n)$ such that $F_i(t_0) \cap A_n \neq \emptyset$ for $i \geq j_0(n)$. We assume that $j_0 \to \infty$ when $n \to \infty$. Exist a sequence $x_n \in F_{j_0(n)}(s_0) \cap A_n$ for $n = 1, 2, ...$.

Put $\lim_{n \to \infty} x_n = x_0$. We have that $x_0 \in A$ becouse $dist(x_n, A) \leq \frac{1}{n}$ and

$$dist(x_n, F(t_0)) \leq \rho_H(F_{j_0(n)}(t_0), F(t_0)) \leq \rho_P(F_{j_0(n)}(t_0), F(t_0)) \to 0$$

so $x_0 \in F(t_0)$, hence $T_A = \bigcap_n T_n$.

The T_n is measurable then T_A is also.

We remark that $G_n(t) = F_n(t) \cap A_n$ for $t \in T_A$ is a sequence of simple multifunctions converges to $F_A(t)$ for almost everywhere $t \in T_A$.

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Certain differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator

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ABSTRACT: In the present paper we define a new operator using the generalized Sălăgean operator and Ruscheweyh operator. Denote by DR_λ^n the Hadamard product of the generalized Sălăgean operator D_λ^n and Ruscheweyh operator R^n, given by $DR_\lambda^n : A \to A$, $DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z)$ and $A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions with $A_1 = A$. We study some differential subordinations regarding the operator DR_λ^n.

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let
$$A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots, z \in U\},$$

for $n \in \mathbb{N}$ and $A_1 = A$.

Denote by
$$K = \left\{ f \in A : \text{Re } \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ for all

$z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \qquad \text{for} \quad z \in U, \qquad (1.1)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfing (1.1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Definition 1 *(Al Oboudi [2]) For $f \in A$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n : A \to A$,*

$$
\begin{aligned}
D_\lambda^0 f(z) &= f(z) \\
D_\lambda^1 f(z) &= (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\
&\quad \dots \\
D_\lambda^n f(z) &= (1 - \lambda) D_\lambda^{n-1} f(z) + \lambda z \left(D_\lambda^n f(z) \right)' = D_\lambda \left(D_\lambda^{n-1} f(z) \right), \quad for \ \ z \in U.
\end{aligned}
$$

Remark 1 *If $f \in A$ and $f(z) = z + \sum_{j=2}^\infty a_j z^j$, then*
$D_\lambda^n f(z) = z + \sum_{j=2}^\infty \left[1 + (j - 1) \lambda \right]^n a_j z^j$, *for $z \in U$.*

Remark 2 *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [5].*

Definition 2 *(Ruscheweyh [4]) For $f \in A$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : A \to A$,*

$$
\begin{aligned}
R^0 f(z) &= f(z) \\
R^1 f(z) &= z f'(z) \\
&\quad \dots \\
(n + 1) R^{n+1} f(z) &= z \left(R^n f(z) \right)' + n R^n f(z), \quad for \ \ z \in U.
\end{aligned}
$$

Remark 3 *If $f \in A$ and $f(z) = z + \sum_{j=2}^\infty a_j z^j$, then*
$R^n f(z) = z + \sum_{j=2}^\infty C_{n+j-1}^n a_j z^j$, *for $z \in U$.*

Lemma 1 *(Miller and Mocanu [3]) Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z), \quad for \ \ z \in U,$$

where $\alpha > 0$ and n is a positive integer.
   If
$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad for \ \ z \in U$$

is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \quad for \ z \in U,$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

2 Main Results

Definition 3 *Let $\lambda \geq 0$ and $n \in \mathbb{N}$. Denote by $DR_\lambda^n : A \to A$ the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator D_λ^n and the Ruscheweyh operator R^n:*

$$DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z),$$

for any $z \in U$ and each nonnegative integer n.

Remark 4 *If $f \in A$ and $f(z) = z + \sum_{j=2}^\infty a_j z^j$, then $DR_\lambda^n f(z) = z + \sum_{j=2}^\infty C_{n+j-1}^n [1 + (j-1)\lambda]^n a_j^2 z^j$, for $z \in U$.*

Remark 5 *For $\lambda = 1$ we obtain the Hadamard product SR^n [1] of the Sălăgean operator S^n and Ruscheweyh operator R^n.*

Theorem 2 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in A$ and the differential subordination*

$$\frac{n+1}{\lambda z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{\lambda z} DR_\lambda^n f(z) - \left(n - 1 + \frac{1}{\lambda}\right) (DR_\lambda^n f(z))' \prec h(z), \quad (2.2)$$

for $z \in U$, holds, then

$$(DR_\lambda^n f(z))' \prec g(z), \quad for \ z \in U,$$

and this result is sharp.

Proof. With notation

$$p(z) = (DR_\lambda^n f(z))' = 1 + \sum_{j=2}^\infty C_{n+j-1}^n [1 + (j-1)\lambda]^n j a_j^2 z^{j-1}$$

and $p(0) = 1$, we obtain for $f(z) = z + \sum_{j=2}^\infty a_j z^j$,

$$p(z) + zp'(z) = 1 + \sum_{j=2}^{\infty} C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n j^2 a_j^2 z^{j-1}$$

$$= \frac{n+1}{\lambda z}\left[z + \sum_{j=2}^{\infty} C_{n+j}^{n+1}\left[1+(j-1)\lambda\right]^{n+1} a_j^2 z^j\right] + \frac{\lambda - n - 1}{\lambda}$$

$$- \sum_{j=2}^{\infty} C_{n+j-1}^n \left[1+(j-1)\lambda\right]^n a_j^2 z^{j-1}\left(n-1+\frac{1}{\lambda}\right)j$$

$$- \sum_{j=2}^{\infty} C_{n+j-1}^n \left[1+(j-1)\lambda\right]^n a_j^2 z^{j-1}\frac{n(1-\lambda)}{\lambda}$$

$$= \frac{n+1}{\lambda z} DR_\lambda^{n+1} f(z) - \left(n-1+\frac{1}{\lambda}\right)(DR_\lambda^n f(z))' - \frac{n(1-\lambda)}{\lambda z} DR_\lambda^n f(z).$$

We have $p(z) + zp'(z) \prec h(z)$, for $z \in U$. By using Lemma 1 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $(DR_\lambda^n f(z))' \prec g(z)$, for $z \in U$ and this result is sharp. ∎

Corollary 3 *(see [1]) Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$, $f \in A$ and the differential subordination*

$$\frac{1}{z}SR^{n+1}f(z) + \frac{n}{n+1}z(SR^n f(z))'' \prec h(z), \quad for \ \ z \in U, \tag{2.3}$$

holds, then

$$(SR^n f(z))' \prec g(z), \quad for \ \ z \in U$$

and this result is sharp.

Theorem 4 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$(DR_\lambda^n f(z))' \prec h(z), \quad for \ \ z \in U, \tag{2.4}$$

then

$$\frac{DR_\lambda^n f(z)}{z} \prec g(z), \quad for \ \ z \in U,$$

and this result is sharp.

Proof. For $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n \left[1+(j-1)\lambda\right]^n a_j^2 z^j \quad for, z \in U$$

Consider

$$p(z) = \frac{DR_\lambda^n f(z)}{z} = \frac{z + \sum_{j=2}^\infty C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n a_j^2 z^j}{z}$$

$$= 1 + \sum_{j=2}^\infty C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n a_j^2 z^{j-1}.$$

We have $p(z) + zp'(z) = (DR_\lambda^n f(z))'$, for $z \in U$.

Then $(DR_\lambda^n f(z))' \prec h(z)$, for $z \in U$, becomes $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, for $z \in U$. By using Lemma 1 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $\frac{DR_\lambda^n f(z)}{z} \prec g(z)$, for $z \in U$. ∎

Corollary 5 *(see [1]) Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$(SR^n f(z))' \prec h(z), \quad for \ z \in U, \tag{2.5}$$

then

$$\frac{SR^n f(z)}{z} \prec g(z), \quad for \ z \in U,$$

and this result is sharp.

Theorem 6 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$\left(\frac{zDR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)}\right)' \prec h(z), \quad for \ z \in U, \tag{2.6}$$

then

$$\frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} \prec g(z), \quad for \ z \in U,$$

and this result is sharp.

Proof. For $f \in A$ and $f(z) = z + \sum_{j=2}^\infty a_j z^j$ we have

$$DR_\lambda^n f(z) = z + \sum_{j=2}^\infty C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n a_j^2 z^j, \text{for} z \in U$$

. Consider

$$p(z) = \frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} = \frac{z + \sum_{j=2}^\infty C_{n+j}^{n+1} \left[1 + (j-1)\lambda\right]^{n+1} a_j^2 z^j}{z + \sum_{j=2}^\infty C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n a_j^2 z^j}$$

$$= \frac{1 + \sum_{j=2}^\infty C_{n+j}^{n+1} \left[1 + (j-1)\lambda\right]^{n+1} a_j^2 z^{j-1}}{1 + \sum_{j=2}^\infty C_{n+j-1}^n \left[1 + (j-1)\lambda\right]^n a_j^2 z^{j-1}}.$$

We have $p'(z) = \frac{\left(DR_\lambda^{n+1}f(z)\right)'}{DR_\lambda^n f(z)} - p(z) \cdot \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}$.

Then $p(z) + zp'(z) = \left(\frac{zDR_\lambda^{n+1}f(z)}{DR_\lambda^n f(z)}\right)'$.

Relation (2.6) becomes $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, for $z \in U$, and, by using Lemma 1 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $\frac{DR_\lambda^{n+1}f(z)}{DR_\lambda^n f(z)} \prec g(z)$, for $z \in U$. ∎

Corollary 7 *(see [1]) Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$\left(\frac{zSR^{n+1}f(z)}{SR^n f(z)}\right)' \prec h(z), \quad for \ \ z \in U, \tag{2.7}$$

then

$$\frac{SR^{n+1}f(z)}{SR^n f(z)} \prec g(z), \quad for \ \ z \in U,$$

and this result is sharp.

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Multiple hypotheses optimal testing for Markov chains and identification subject to the reliability criterion

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ABSTRACT: The problem of many ($L > 2$) hypotheses testing on distributions of a finite state Markov chain is studied. We apply large deviation techniques (LDT). It is demonstrated that this method of investigation in solving the problem of logarithmically asymptotically optimal (LAO) hypotheses testing is easier, compared with the procedure introduced by Haroutunian. The matrix of exponents $\mathbf{E} = \{E_{l|m}\}$, $m, l = \{1, 2, \ldots, L\}$, of error probabilities of the LAO test $E_{l|m}(\phi) = \lim\limits_{N \to \infty} -\frac{1}{N} \log \alpha_{l|m}^{(N)}(\phi_N)$, where $\alpha_{l|m}^{(N)}(\phi_N)$ for $l \neq m$ is the probability to accept the hypothesis l, when the hypothesis m is true, is determined.

Moreover, the identification of distributions for one object and two independent objects via simple homogeneous stationary Markov chains with finite number of states is discussed in the present paper.

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1. Introduction

Applications of information-theoretical methods in mathematical statistics are illustrated in the monographs presented by Kullback [9], Csiszár and Körner [4], Blahut [2], Csiszár and Shields [3], Gutman [6] and others. Numerous papers have been devoted to the study of exponential decrease, as the sample size N goes to infinity, of the error probabilities $\alpha_1^{(N)}$ of the first kind and $\alpha_2^{(N)}$ of the second kind of the optimal tests for two simple statistical hypotheses. Similar problems for Markov dependence

of experiments were investigated by Natarajan [10], Haroutunian [7], [8], Dembo and Zeitouni [5], and others. In the book of Csiszár and Shields [3] different asymptotic aspects of two hypotheses testing for independent identically distributed observations are considered via theory of large deviations.

In this paper, we aim to solve the problem in order to describe the matrix of exponents $E = \{E_{l|m}\}, m, l = \{1, 2, \ldots, L\}$ of probabilities $\alpha_{l|m}^{(N)} = \exp(-NE_{l|m})$, where $\alpha_{l|m}^{(N)}$ for $l \neq m$ is the probability to accept hypothesis l, when hypothesis m is true, for finite state Markov chain by application of large deviation techniques (LDT). We will demonstrated that the solution of the mentioned problem is more concise and hence easier than the procedure introduced by Haroutunian [7].

Ahlswede and Haroutunian [1], formulated an ensemble of problems on multiple hypotheses testing for multiple objects and on identification of hypotheses under reliability requirement. In this paper, we also solve this problem through identification of distributions of many hypotheses for one object and two independent objects, using simple homogeneous stationary finite states of Markov chains.

In Section 2, we present a Theorem of LDT for Markov chains and the result for hypotheses testing and in Section 3, the problem of identification for Markov chain and finally in Section 4, we discuss the general case of the problem of identification of distributions for two independent Markov chains.

2. Application of LDT On Many Hypotheses Optimal Testing for Markov chains

Let $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_N)$, $x_n \in \mathcal{X} = \{1, 2, \ldots, I\}$, $\mathbf{x} \in \mathcal{X}^{N+1}$, $N = 0, 1, 2, \ldots$, be vectors of observations of a simple homogeneous stationary Markov chain with finite number I of states. The hypotheses concern the irreducible matrices of the transition probabilities

$$P_l = \{P_l(j|i),\, i, j = \{1, 2, \ldots, I\}\}, l = \{1, 2, \ldots, L\}.$$

The stationarity of the chain provides existence for each $l = \{1, 2, \ldots, L\}$ of the unique stationary distribution $Q_l = \{Q_l(i),\; i = \{1, 2, \ldots, I\}$, such that

$$\sum_i Q_l(i) P_l(j|i) = Q_l(j), \quad \sum_i Q_l(i) = 1, \;\; i, j = \{1, 2, \ldots, I\}.$$

We define the joint distributions

$$Q_l \circ P_l = \{Q_l(i) P_l(j|i), i, j = \{1, 2, \ldots, I\}\}, l = \{1, 2, \ldots, L\}.$$

Let us denote $D(Q \circ P \| Q_l \circ P_l)$ Kullback-Leibler divergence(P is an irreducible matrix of transition probabilities of some sttionary Markov chain and Q be the corresponding stationary PD)

$$D(Q \circ P \| Q_l \circ P_l) = \sum_{i,j} Q(i) P(j|i) [\log Q(i) P(j|i) - \log Q_l(i) P_l(j|i)]$$

$$= D(Q\|Q_l) + D(Q \circ P\|Q \circ P_l),$$

of distribution

$$Q \circ P = \{Q(i)P(j|i), i, j = \{1, 2, \dots, I\}\},$$

with respect to distribution $Q_l \circ P_l$ where

$$D(Q\|Q_l) = \sum_i Q(i)[\log Q(i) - \log Q_l(i)], \ l = \{1, 2, \dots, L\}.$$

Let us name the second order type of vector $\mathbf{x}$ the square matrix of I^2 relative frequencies $\{N(i,j)N^{-1}, i, j = \{1, 2, \dots, I\}\}$ of the simultaneous appearance in $\mathbf{x}$ of the states i and j on the pairs of neighbor places. It is clear that $\sum_{ij} N(i,j) = N$. Denoted by $\mathcal{T}_{Q \circ P}^N$, the set of vectors from $\mathcal{X}^{N+1}$ have the second order type in a way that for some joint PD $Q \circ P$

$$N(i,j) = NQ(i)P(j|i), \quad i = \{1, 2, \dots, I\}, \quad j = \{1, 2, \dots, I\}.$$

The set of all joint PD $Q \circ P$ on $\mathcal{X}$ is denoted by $\mathcal{Q} \circ \mathcal{P}(\mathcal{X})$ and the set of all possible second order types for joint PD $Q \circ P$ is denoted by $\mathcal{Q} \circ \mathcal{P}^N(\mathcal{X})$. Note that if vector $\mathbf{x} \in \mathcal{T}_{Q \circ P}^N$, then

$$\sum_j N(i,j) = NQ(i), \ i = \{1, 2, \dots, I\}, \quad \sum_i N(i,j) = NQ'(j), \ j = \{1, 2, \dots, I\},$$

for somewhat different from Q PD Q', which in accordance with the definition of $N(i,j)$, are closed enough

$$|NQ(i) - NQ'(i)| \leq 1, \ i = \{1, 2, \dots, I\},$$

and in the limit, when $N \to \infty$, the distribution Q coincides with Q' and may be taken as stationary for conditional PD P:

$$\sum_i Q(i)P(j|i) = Q(j), \ j \in \mathcal{X}.$$

The probability of vector $\mathbf{x} \in \mathcal{X}^{N+1}$ of the Markov chain with transition probabilities P_l and stationary distribution Q_l, is the following

$$Q_l \circ P_l^N(\mathbf{x}) = Q_l(x_0) \prod_{n=1}^N P_l(x_n|x_{n-1}), l = \{1, 2, \dots, I\},$$

$$Q_l \circ P_l^N(\mathcal{A}) = \bigcup_{\mathbf{x} \in \mathcal{A}} Q_l \circ P_l^N(\mathbf{x}), \ \mathcal{A} \subset \mathcal{X}^{N+1}.$$

Note that for $l = \{1, 2, \dots, L\}$ the probability of $\mathbf{x}$ from $\mathcal{T}_{Q \circ P}^N$ can be written as

$$Q_l \circ P_l^N(\mathbf{x}) = Q_l(x_0) \prod_{i,j} P_l(j|i)^{NQ(i)P(j|i)}.$$

Note also that if $Q \circ P$ is absolutely continuous relative to $Q_l \circ P_l$, then

$$Q_l \circ P_l^N(\mathcal{T}_{Q \circ P}^N) = \exp\{-N(D(Q \circ P \| Q \circ P_l)) + o(1)\},$$

where

$$o(1) = \max(\max_i |N^{-1} \log Q_l(i)| : Q_l(i) > 0),$$

$$(\max_i |N^{-1} \log Q_l(i)| : Q_l(i) > 0) \to 0, \quad \text{when } N \to \infty.$$

Indeed, this is not difficult to verify, taking into account the number $|\mathcal{T}_{Q \circ P}^N|$ of vectors in $\mathcal{T}_{Q \circ P}^N$ which is equal to

$$\exp\{-N(\sum_{i,j} Q(i)P(j|i) \log P(j|i)) + o(1)\}.$$

Non-randomized test $\phi_N(\mathbf{x})$ accepts one of the hypotheses H_l, $l = \{1, 2, \ldots, L\}$ on the basis of the trajectory $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ of the $N + 1$ observations. Let us denote $\alpha_{l|m}^{(N)}(\phi_N)$ the probability to accept the hypothesis H_l under the condition that H_m, $m \neq l$, is true. For $l = m$ we denote $\alpha_{m|m}^{(N)}(\phi_N)$ the probability to reject the hypothesis H_m. It is clear that

$$\alpha_{m|m}^{(N)}(\phi_N) = \sum_{l \neq m} \alpha_{l|m}^{(N)}(\phi_N), \ m = \{1, 2, \ldots, L\}. \tag{1}$$

This probability is called the error probability of the m-th kind of the test ϕ_N. The quadratic matrix of L^2 error probabilities $\{\alpha_{l|m}^{(N)}(\phi), \ m, l = \{1, 2, \ldots, L\}$ is occasionally called the power of the tests. To every trajectory $\mathbf{x}$, the test ϕ_N puts in one correspondence from L hypotheses. Thus, the space $\mathcal{X}^{N+1}$ will be divided into L parts,

$$\mathcal{G}_l^N = \{\mathbf{x}, \ \phi_N(\mathbf{x}) = l\}, \ l = \{1, 2, \ldots, L\},$$

and

$$\alpha_{l|m}^N(\phi_N) = Q_m \circ P_m(\mathcal{G}_l^N), \quad m, l = \{1, 2, \ldots, L\}.$$

We study the matrix of "reliabilities",

$$E_{l|m}(\phi) = \overline{\lim_{N \to \infty}} - \frac{1}{N} \log \alpha_{l|m}^{(N)}(\phi_N), \ m, l = \{1, 2, \ldots, L\}. \tag{2}$$

Note that from definitions (1) and (2) it follows that:

$$E_{m|m} = \min_{l \neq m} E_{l|m}. \tag{3}$$

Definition. *The test sequence $\Phi^* = (\phi_1, \phi_2, \ldots)$ is called LAO if for a given family of positive numbers $E_{1|1}, E_{2|2}, \ldots, E_{L-1|L-1}$, the reliability matrix contains in the diagonal these numbers and the remained $L^2 - L + 1$ components take the maximal possible values.*

Let $P = \{P(j|i)\}$ be a irreducible matrix of transition probabilities of some stationary Markov chain with the same set $\mathcal{X}$ of states, and $Q = \{Q(i), i = \{1, 2, \ldots, I\}$ be the corresponding stationary PD.

For a given family of positive numbers $E_{1|1}, E_{2|2}, \ldots, E_{L-1|L-1}$, let us define the decision rule ϕ^* by the following sets

$$
\begin{aligned}
\mathcal{R}_l &= \{Q \circ P : D(Q \circ P \| Q \circ P_l) \leq E_{l|l}, \ D(Q \| Q_l) < \infty\}, \ l = \{1, 2, \ldots, L-1\}, \\
\mathcal{R}_L &= \{Q \circ P : D(Q \circ P \| Q \circ P_l) > E_{l|l}, \ l = \{1, 2, \ldots, L-1\}, \quad\quad (4) \\
\mathcal{R}_l^N &= \mathcal{R}_l \cap \mathcal{Q} \circ \mathcal{P}^N(\mathcal{X}), \quad\quad l = \{1, 2, \ldots, L\}.
\end{aligned}
$$

and introduce the functions:

$$
\begin{aligned}
E_{l|l}^*(E_{l|l}) &= E_{l|l}, \ l = \{1, 2, \ldots, L-1\} \quad\quad (5) \\
E_{l|m}^*(E_{l|l}) &= \inf_{Q \circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m), \ m = \{1, \ldots, L\}, l \neq m, \ l = \{1, \ldots, L-1\}, \\
E_{L|m}^*(E_{1|1}, \ldots, E_{L-1|L-1}) &= \inf_{Q \circ P \in \mathcal{R}_L} D(Q \circ P \| Q \circ P_m), \ m = \{1, 2, \ldots, L-1\}, \\
E_{L|L}^*(E_{1|1}, \ldots, E_{L-1|L-1}) &= \min_{l = \{1, 2, \ldots, L-1\}} E_{l|L}^*.
\end{aligned}
$$

We cite the statement of the general case of large deviation result for types by Natarajan [10].

Theorem 1. *Let $\mathcal{X} = \{1, 2, \ldots, I\}$ be a finite set of the states of the stationary Markov chain, possessing an irreducible transition matrix P and $\mathcal{A}$ be a nonempty and open subset or convex subset of joint distributions $Q \circ P$ and Q_m is stationary distribution for P_m, then for the type $Q \circ P(\mathbf{x})$ of a vector $\mathbf{x}$ from $Q_m \circ P_m$ on $\mathcal{X}$:*

$$
\lim_{N \to \infty} -\frac{1}{N} \log Q_m \circ P_m^N \{\mathbf{x} : Q \circ P(\mathbf{x}) \in \mathcal{A}\} = \inf_{Q \circ P \in \mathcal{A}} D(Q \circ P \| Q \circ P_m).
$$

Now we formulate the theorem from [7], which we prove by application of Theorem 1.

Theorem 2. *Let $\mathcal{X}$ be a fixed finite set, for a family of distinct distributions $P_1, \ldots, P_L$ the following two statements hold. If the positive finite numbers $E_{1|1}, \ldots, E_{L-1|L-1}$ satisfy conditions:*

$$
0 < \ E_{1|1} \ < \min\left[D(Q_m \circ P_m \| Q_m \circ P_1), \ m = \{2, \ldots, L\}\right], \quad\quad (6)
$$

$$
\ldots
$$

$$
0 < \ E_{l|l} \ < \min\left[\begin{array}{l} E_{l|m}^*(E_{m|m}), \ m = \{1, 2, \ldots, l-1\}, \\ D(Q_m \circ P_m \| Q_m \circ P_l), \ m = \{l+1, \ldots, L\} \end{array} \right], l = \{2, \ldots, L-1\},
$$

then:

a) there exists a LAO sequence of tests ϕ^*, the reliability matrix of which $\left\{E_{l|m}^*(\phi^*)\right\}$ is defined in (5), and all elements of it are positive,

b) even if one of conditions (6) is violated, then the reliability matrix of an arbitrary test necessarily has an element equal to zero, (the corresponding error probability does not tend exponentially to zero).

Proof: First we remark that $D(Q \circ P_l \| Q \circ P_m) > 0$, for $l \neq m$, because all measures $P_l, l = \{1, 2, \ldots, L\}$, are distinct. Let us prove the statement a) of the theorem 2 about the existence of the sequence corresponding to a given $E_{1|1}, \cdots, E_{L-1|L-1}$ satisfying condition (6). Consider the following sequence of tests ϕ^* given by the sets

$$\mathcal{B}_l^N = \bigcup_{Q \circ P \in \mathcal{R}_l^N} \mathcal{T}_{Q \circ P}^N(\mathbf{x}), l = \{1, 2, \ldots, L\}. \tag{7}$$

Notice that on account of condition (6) and the continuity of divergence D for N large enough the sets $\mathcal{R}_l^N, l = \{1, 2, \ldots, L\}$ from (4) are not empty. The sets $\mathcal{B}_l^N, l = \{1, 2, \ldots, L\}$, satisfy conditions :

$$\mathcal{B}_l^N \bigcap \mathcal{B}_m^N = \emptyset, l \neq m, \qquad \bigcup_{l=1}^{L} \mathcal{B}_l^N = \mathcal{X}^N.$$

Now let us demonstrate that, exponent $E_{l|m}(\phi^*)$ for sequence of tests ϕ^* defined in (7) is equal to $E_{l|m}^*$. We know from (4) that $\mathcal{R}_l, l = \{1, 2, \ldots, L-1\}$, are convex subset and $\mathcal{R}_L$ is open subset of the decision rule of ϕ^*, therefore $\mathcal{R}_l, \quad l = \{1, 2, \ldots, L\}$, satisfy in condition of Theorem 1. With relations (4), (5), by Theorem 1 we have

$$\lim_{N \to \infty} -\frac{1}{N} \log \alpha_{l|m}^N(\phi^*) = \lim_{N \to \infty} -\frac{1}{N} \log Q \circ P_m^N(\mathcal{R}_l) = \inf_{Q \circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m).$$

Now using (2) we obtain the following:

$$E_{l|m}(\phi^*) = \inf_{Q \circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m) \quad m, l = \{1, 2, \ldots, L\}.$$

Using (6), (4) and (5) it can be realized that all $E_{l|m}^*$ are strictly positive. The proof of part (a) will be concluded if one demonstrates that the sequence of the tests ϕ^* is LAO, that is at given finite $E_{1|1}, \cdots, E_{L-1|L-1}$ for any other sequence of tests ϕ^{**}

$$E_{l|m}^*(\phi^{**}) \leq E_{l|m}^*(\phi^*), \quad m, l = \{1, 2, \ldots, L\}.$$

For this purpose it is sufficient to demonstrate that the sequence of tests will not asymptotically become better if the sets $\mathcal{R}_l^N$ are not a union of some number of whole types $\mathcal{T}_{Q \circ P}^N(\mathbf{x})$, indeed, if a test ϕ^{**} is defined, for instance, by sets $\mathcal{G}_1^N, \cdots, \mathcal{G}_L^N$ and, furthermore, $Q \circ P$ is such that

$$0 \neq Q_l \circ P_l^N(\mathcal{G}_l^N \bigcap \mathcal{T}_{Q \circ P}^N(\mathbf{x})) = Q_l \circ P_l^N(\mathcal{T}_{Q \circ P}^N(\mathbf{x})) + o(1),$$

Then, the test ϕ^{**} will be improved, if instead of the set $\mathcal{G}_l^N$ one takes $\mathcal{G}_l^N \bigcup \mathcal{T}_{Q \circ P}^N(\mathbf{x})$, as the error probability $\alpha_{l|m}^N$ can decrease for $m \neq l$. The statement of part (b) of theorem is evident, since the violation of one of the conditions (6) reduces to the equality to zero of at least one of the elements $E_{l|m}^*$ defined in (5).

3. On Statistical Identification of Markov Chain of Distribution Subject to the reliability

Assume that there are $L \geq 2$ hypothetical distributions. The question here is whether or not r-th distribution has occurred.

There are two error probabilities for each $r = \{1, 2, \ldots, L\}$, the probability $\alpha_{l \neq r | m = r}^{(N)}$ to accept l different from r, when r is in reality, and the probability $\alpha_{l = r | m \neq r}^{(N)}$ that r is accepted, when it is not correct. The probability $\alpha_{l \neq r | m = r}^{(N)}$ we already know, it is the probability $\alpha_{r|r}^{(N)}$ which is equal to $\sum_{l:l \neq r} \alpha_{l|r}^{(N)}$. The reliability $E_{l \neq r | m = r}$ coincides with $E_{r|r}$, with (3). Our aim is to find the interdependence between $E_{l \neq r | m = r}$ and $E_{m \neq r | l = r}$. The latter, can have values satisfying in conditions (6), thus, we will have the following conditions:

$$0 < E_{r|r} < \min_{l:l \neq r}[D(Q_l \circ P_l \| Q_l \circ P_r), \ r = \{1, 2, \ldots, L\}.$$

We need to use the probabilities of different hypotheses. Let us assume that the hypotheses $H_l, l = \{1, 2, \ldots, L\}$, have positive probabilities say $P_{\mathbf{r}}(r), r = \{1, 2, \ldots, L\}$. We will see that the formulated result in the following theorem, does not depend on values of $P_{\mathbf{r}}(r), r = \{1, 2, \ldots, L\}$, if they all are strictly positive. Thus we can make the following calculations for $r = \{1, 2, \ldots, L\}$:

$$\alpha_{l=r|m\neq r}^{(N)} = \frac{P_{\mathbf{r}}^{(N)}(l = r, m \neq r)}{P_{\mathbf{r}}(m \neq r)} = \frac{1}{\sum_{m:m\neq r} P_{\mathbf{r}}(m)} \sum_{m:m\neq r} P_{\mathbf{r}}^{(N)} \alpha_{r|m}^{(N)},$$

and also for $r = \{1, 2, \ldots, L\}$, we obtain the following:

$$\begin{aligned} E_{l=r|m\neq r} &= \overline{\lim_{N\to\infty}}(-\frac{1}{N} \log \alpha_{l=r|m\neq r}^{(N)}) \qquad\qquad (8) \\ &= \overline{\lim_{N\to\infty}} \frac{1}{N}(\log \sum_{m:m\neq r} P_{\mathbf{r}}(m) - \log \sum_{m:m\neq r} P_{\mathbf{r}}^{(N)} \alpha_{r|m}^{(N)}) = \min_{m:m\neq r} E_{r|m}^*. \end{aligned}$$

Using (8) by analogy with Theorem 1 and Theorem 2, we conclude (with $\mathcal{R}_r$ as in (4) for each r including $r = L$ by the values of $E_{r|r}$ from $(0, \min D(Q_l \circ P_l \| Q_l \circ P_r)))$, that

$$\begin{aligned} E_{l=r|m\neq r}(E_{r|r}) &= \min_{m:m\neq r} \ \inf_{Q\circ P\in\mathcal{R}_r} D(Q \circ P \| Q \circ P_m) \qquad\qquad (9) \\ &= \min_{m:m\neq r} \ \inf_{Q\circ P:D(Q\circ P\|Q\circ P_r)\leq E_{r|r}} D(Q \circ P \| Q \circ P_m), r = \{1, 2, \ldots, L\}. \end{aligned}$$

Thus, the obtained result may be formulate in the following theorem.

Theorem 3. *For the model with distinct distributions for the given sample* $\mathbf{x}$*, we can determine its type* $Q \circ P$*, and when* $Q \circ P \in \mathcal{R}_r^{(N)}$*, we accept the hypotheses* r*. Under the condition that the probabilities of all* L *hypotheses are positive, the reliability of such test* $E_{l=r|m\neq r}$ *for given* $E_{l\neq r|m=r}$ *is defined in (9).*

4. On Identification of Two Independent Markov chain of Distributions

In this section, we expand the concept of section 3 for two independent homogenies stationary finite Markov chain. Let x_1 and x_2 be independent RV, taking values in the same finite state of Markov chain of set $\mathcal{X}$ with one of L PDs, being characteristics of corresponding independent objects, the random vector (X_1, X_2) assume values $(x^1, x^2) \in \mathcal{X} \times \mathcal{X}$.

Let
$(\mathbf{x_1}, \mathbf{x_2}) = ((x_0^1, x_0^2), \ldots, (x_n^1, x_n^2), \ldots, (x_N^1, x_N^2)), x^i \in \mathcal{X}, i = 1, 2, n = \{0, 2, \ldots, N\},$
be a sequence of results of $N + 1$ independent observations of a simple homogeneses stationary Markov chain with finite number I of states . The statistication must define unknown PDs of the objects on the basis of observed data. The selection for each object was done and it was denoted by Φ_N. The objects independence test Φ_N may be considered as the pair of tests φ_N^1 and φ_N^2 for the respective separate objects. We will show the whole compound test sequence by Φ. The test φ_N^i is defined by a partition of the space $\mathcal{X}^{N+1}$ on the L sets and to every trajectory $\mathbf{x}$ the test ϕ_N puts in one correspondence from L hypotheses. So the space $\mathcal{X}^{N+1}$ will be divided into L parts,

$$\mathcal{G}_{l,i}^N = \{\mathbf{x_i}, \ \phi_N(\mathbf{x_i}) = l\}, \ l = \{1, 2, \ldots, L\}, i = 1, 2.$$

We define

$$\alpha_{l_1, l_2 | m_1, m_2}(\Phi_N) = Q_{m_1} \circ P_{m_1}(\mathcal{G}_{l_1, 1}^N) Q_{m_2} \circ P_{m_2}(\mathcal{G}_{l_2, 2}^N),$$

as the probability of the erroneous acceptance by the test Φ_N of the hypotheses pair (H_{l_1}, H_{l_2}), provided that (H_{m_1}, H_{m_2}) is true, where $(m_1, m_2) \neq (l_1, l_2), m_i, l_i = \{1, 2, \ldots, L\}, i = 1, 2$. The probability to reject a true pair of hypotheses (H_{m_1}, H_{m_2}) by analogy with(1) is the following:

$$\alpha_{m_1, m_2 | m_1, m_2}^N(\Phi_N) \triangleq \sum_{(l_1, l_2) \neq (m_1, m_2)} \alpha_{l_1, l_2 | m_1, m_2}^N(\Phi_N). \tag{10}$$

We also study corresponding limits $E_{l_1, l_2 | m_1, m_2}(\Phi_N)$ of error probability exponents of the sequence of tests Φ, called reliabilities :

$$E_{l_1, l_2 | m_1, m_2}(\Phi) \triangleq \varliminf_{N \to \infty} - \frac{1}{N} \log \alpha_{l_1, l_2 | m_1, m_2}(\Phi_N), \quad m_i, l_i = \overline{1, L}, \quad i = 1, 2. \tag{11}$$

We denote by $E(\varphi^i)$ the reliability matrices of the sequences of tests $\varphi^i, i = 1, 2$, for each of the objects.

Applying (10) and (11), we obtain the following :

$$E_{m_1, m_2 | m_1, m_2}(\Phi) = \min_{(l_1, l_2) \neq (m_1, m_2)} E_{l_1, l_2 | m_1, m_2}(\Phi).$$

In this section we use the following lemma.

Lemma. [8], [11] If elements $E_{l|m}(\varphi^i), m, l = \{1, 2, \ldots, L\}, i = 1, 2$, are strictly positive, then the following equalities hold for $\Phi = (\varphi^1, \varphi^2)$:

$$E_{l_1, l_2|m_1, m_2}(\Phi) = E_{l_1|m_1}(\varphi^1) + E_{l_2|m_2}(\varphi^2), \quad if \quad m_1 \neq l_1, \quad m_2 \neq l_2, \qquad (12a)$$

$$E_{l_1, l_2|m_1, m_2}(\Phi) = E_{l_i|m_i}(\varphi^i), \quad if \quad m_{3-i} = l_{3-i} \quad m_i \neq l_i, \quad i = 1, 2. \qquad (12b)$$

Consider for given positive elements $E_{m,m|m,L}$ and $E_{m,m|L,m}, m = \{1, 2, \ldots, L - 1\}$, the family of regions:

$$\mathcal{R}_m^{(1)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) \leq E_{m,m|L,m}\}, \quad m = \{1, 2, \ldots, L - 1\},$$

$$\mathcal{R}_m^{(2)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) \leq E_{m,m|m,L}\}, \quad m = \{1, 2, \ldots, L - 1\},$$

$$\mathcal{R}_L^{(1)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) > E_{m,m|L,m}, \quad m = \{1, 2, \ldots, L - 1\},$$

$$\mathcal{R}_L^{(2)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) > E_{m,m|m,L}, \quad m = \{1, 2, \ldots, L - 1\}.$$

What is the identification of the probability distributions for two independent objects? The answer for this question constitutes a reply of the question whether or not the pair of the pair of distributions (r_1, r_2) have occurred.

There are two error probabilities for each $(r_1, r_2), r_i = \{1, 2, \ldots, L\}, i = 1, 2$, the probability $\alpha_{(l_1, l_2) \neq (r_1, r_2)|(m_1, m_2) = (r_1, r_2)}^{(N)}$ to accept (l_1, l_2) different from (r_1, r_2), when (r_1, r_2) is in reality, and the probability $\alpha_{(l_1, l_2) = (r_1, r_2)|(m_1, m_2) \neq (r_1, r_2)}^{(N)}$ that (r_1, r_2) is accepted, when it is not correct. The probability $\alpha_{(l_1, l_2) \neq (r_1, r_2)|(m_1, m_2) = (r_1, r_2)}^{(N)}$ is already known, it coincides with the probability $\alpha_{(r_1, r_2)|(r_1, r_2)}^{(N)}$. Our aim is to determine the dependence of $\alpha_{(l_1, l_2) = (r_1, r_2)|(m_1, m_2) \neq (r_1, r_2)}^{(N)}$ on given $\alpha_{(r_1, r_2)|(r_1, r_2)}^{(N)}$.

We need to use the probabilities of different hypotheses. Let us assume that the hypotheses $H_l : l = \{1, 2, \ldots, L\}$ have, say, probabilities $P_\mathbf{r}(r), r = \{1, 2, \ldots, L\}$. The only supposition we shall use is that $P_\mathbf{r}(r) > 0, r = \{1, 2, \ldots, L\}$. We demonstrate, the result formulated in the following theorem does not depend on values of $P_\mathbf{r}(r), r = \{1, 2, \ldots, L\}$, if they all are strictly positive. Thus, the following reasoning can be made for each $r_i = \{1, 2, \ldots, L\}, i = 1, 2$:

$$\alpha_{(l_1, l_2) = (r_1, r_2)|(m_1, m_2) \neq (r_1, r_2)}^{(N)} = \frac{P_\mathbf{r}^{(N)}((l_1, l_2) = (r_1, r_2), (m_1, m_2) \neq (r_1, r_2))}{P_\mathbf{r}(m_1, m_2) \neq (r_1, r_2))},$$

$$\alpha_{(l_1, l_2) = (r_1, r_2)|(m_1, m_2) \neq (r_1, r_2)}^{(N)}$$

$$= \frac{1}{\sum\limits_{m : (m_1, m_2) \neq (r_1, r_2)} P_\mathbf{r}(m_1, m_2)} \sum\limits_{m : (m_1, m_2) \neq (r_1, r_2)} \alpha_{(m_1, m_2)|(r_1, r_2)} P_\mathbf{r}^{(N)}(m_1, m_2).$$

Finally, for $r = \overline{1, L}$, we obtain the following:

$$E_{(l_1, l_2) = (r_1, r_2)|(m_1, m_2) \neq (r_1, r_2)} = \min_{(m_1, m_2) : (m_1, m_2) \neq (r_1, r_2)} E_{(r_1, r_2)|(m_1, m_2)}^*. \qquad (13)$$

For every LAO test Φ^* from (11), (12) and (13) we obtain the following:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \min_{m_1\neq r_1,m_2\neq r_2}(E^1_{r_1|m_1}, E^2_{r_2|m_2}),$$

where, $E^1_{r_1|m_1}, E^2_{r_2|m_2}$ are determined by (5) for, correspondingly, the first and the second objects. For every LAO test Φ^* from (11) and (12) we deduce that

$$E_{(r_1,r_2)|(r_1,r_2)} = \min_{m_1\neq r_1,m_2\neq r_2}(E^1_{r_1|m_1}, E^2_{r_2|m_2}) = \min(E^1_{r_1|r_1}, E^2_{r_2|r_2}). \qquad (14)$$

and each of $E^1_{r_1|r_1}, E^2_{r_2|r_2}$ satisfy the following conditions (see theorem 2, condition (6)).

$$0 < E^1_{r_1|r_1} < \min\left[\min_{l=\{1,\ldots,r_1-1\}} E^*_{l|m}(E^1_{l|l}), \min_{l=\{r_1+1,\ldots,L\}} D(Q_l \circ P_l \| Q_l \circ P_{r_1})\right], \quad (15a)$$

$$0 < E^2_{r_2|r_2} < \min\left[\min_{l=\{1,2,\ldots,r_2-1\}} E^*_{l|m}(E^2_{l|l}), \min_{l=\{r_2+1,\ldots,L\}} D(Q_l \circ P_l \| Q_l \circ P_{r_2})\right], \quad (15b)$$

From (5), we see that the elements $E^*_{l|m}(E^1_{l|l}), r_1 = \{1, 2, \ldots, r_1-1\}$ and $E^*_{l|m}(E^2_{l|l}), r_2 = \{1, 2, \ldots, r_2 - 1\}$ are determined by only $E^1_{l|l}$ and $E^2_{l|l}$. However, we are considering only elements $E^1_{r_1|r_1}$ and $E^2_{r_2|r_2}$. Using theorem 1 and (15) we have

$$0 < E^1_{r_1|r_1}$$

$$< \min\left[\min_{l=\{1,2,\ldots,r_1-1\}} D(Q_l \circ P_l \| Q_l \circ P_{r_1}), \min_{l=\{r_1+1,\ldots,L\}} D(Q_l \circ P_l \| Q_l \circ P_{r_1})\right], \quad (16a)$$

$$0 < E^2_{r_2|r_2}$$

$$< \min\left[\min_{l=\{1,2,\ldots,r_2-1\}} D(Q_l \circ P_l \| Q_l \circ P_{r_2}), \min_{l=\{r_2+1,\ldots,L\}} D(Q_l \circ P_l \| Q_l \circ P_{r_2})\right]. \quad (16b)$$

Let us denote $r = \max(r_1, r_2)$ and $k = \min(r_1, r_2)$. From (14) we have that, when $E_{(r_1,r_2)|(r_1,r_2)} = E^1_{r_1|r_1}$, then $E^1_{r_1|r_1} \leq E^2_{r_2|r_2}$ and when $E_{(r_1,r_2)|(r_1,r_2)} = E^2_{r_2|r_2}$, then $E^1_{r_1|r_1} \geq E^2_{r_2|r_2}$. Thus, it can be implied that given strictly positive elements $E_{(r_1,r_2)|(r_1,r_2)}$ must meet both inequalities (16) and the combination of these restrictions gives

$$0 < E_{(r_1,r_2)|(r_1,r_2)}$$

$$< \min\left[\min_{l=\{1,2,\ldots,r-1\}} D(Q_l \circ P_l \| Q_l \circ P_r), \min_{l=\{r+1,\ldots,L\}} D(Q_l \circ P_l \| Q_l \circ P_k)\right]. \quad (17)$$

Using (15) we can determine reliability $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ in function of $E_{(r_1,r_2)|(r_1,r_2)}$ as follows:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}(E_{(r_1,r_2)|(r_1,r_2)})$$

$$= \min_{m_1\neq r_1, m_2\neq r_2}(E_{r_1|r_1}(E_{(r_1,r_2)|(r_1,r_2)}), E_{r_2|r_2}(E_{(r_1,r_2)|(r_1,r_2)}), \qquad (18)$$

where, $(E_{r_1|r_1}(E_{(r_1,r_2)|(r_1,r_2)})$ and $E_{r_2|r_2}(E_{(r_1,r_2)|(r_1,r_2)})$ are determined by (5). The obtained results can be summarized in the following theorem:

Theorem 4. *If the distributions H_m, $m = \{1, 2, \ldots, , L\}$, are different and the given strictly positive numbers $E_{(r_1,r_2)|(r_1,r_2)}$ satisfies condition (17), then the reliability $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ is defined in (18).*

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On intuitionistic fuzzy ideals in Γ - rings

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ABSTRACT: In this paper, we study some properties of intuitionistic
fuzzy ideals of a Γ - ring and prove some results on these.

1   Introduction

The notion of a fuzzy set was introduced by L.A.Zadeh [12], and since then this
concept has been applied to various algebraic structures. The idea of "Intuitionistic
Fuzzy Set" was first introduced by K.T.Atanassov [1] as a generalization of the notion
of fuzzy set. N.Nobusawa [10] introduced the notion of a Γ - ring, as more general than
a ring. W.E.Barnes[2] weakened slightly the conditions in the definition of the Γ - rings
in the sense of Nobusawa. W.E.Barnes [2], S.Kyuno [7,8] and J.Luh [9] studied the
structure of Γ - rings and obtained various generalizations analogous to corresponding
parts in ring theory. Y.B.Jun and C.Y.Lee [5] introduced the concept of fuzzy ideals
in the theory of Γ- rings. In this paper, we study the notion of intuitionistic fuzzy
ideals in Γ - rings and prove some of its properties.

2   Preliminaries

In this section the definition of Γ-ring in the sense of Nobusawa and Barnes is discussed
with examples. Also we include some elementary concepts that are necessary for
this paper. Professor N.Nobusawa introduced the concept of Γ-ring and Professor
W.Barnes generalized this concept.

Definition 2.1[2]. If $M = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma \cdots\}$ be two additive
abelian groups and for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are
satisfied:

1. $x\alpha y \in M$,

2. $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)y = x\alpha y + x\beta y, x\alpha(y + z) = x\alpha y + x\alpha z$,

3. $(x\alpha y)\beta z = x\alpha(y\beta z)$, then M is called a Γ - ring.

If these conditions are strengthened to

(1′) $x\alpha y \in M$, $\alpha x\beta \in \Gamma$,

(2′) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)y = x\alpha y + x\beta y, x\alpha(y + z) = x\alpha y + x\alpha z$,

(3′) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$,

(4′) $x\alpha y = 0$ for all $x, y \in M$ implies $\alpha = 0$,

we then have a Γ - ring in the sense of Nobusawa [10]. As indicated in [10], an example of a Γ - ring is obtained by letting X and Y be abelian groups, $M = Hom(X, Y)$, $\Gamma = Hom(Y, X)$ and $x\alpha y$ be the usual composite map. (While Nobusawa does not explicitly require that M and Γ be abelian groups, it appears clear that this is intended.) We may note that it follows from (1) - (3) that $0\alpha y = x0y = x\alpha 0 = 0$ for all $x, y \in M$ and all $\alpha \in \Gamma$.

Example 2.2. If G and G' are two additive abelian groups, M = Hom(G,G'), Γ = Hom(G',G) then M is a Γ-ring with respect to point wise addition and composition of mappings.

Example 2.3. If U and V be vector spaces over the same field F, M = Hom(U,V), Γ = Hom(V,U). Then M is a Γ-ring with respect to point wise addition and composition of mappings.

Definition 2.4[2]. A subset A of a Γ - ring M is a left (resp. right) ideal of M if A is an additive subgroup of M such that $M\Gamma A \subseteq A$ (resp.$A\Gamma M \subseteq A$), where $M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}$ and $A\Gamma M = \{y\alpha x | y \in A, \alpha \in \Gamma, x \in M\}$. If A is both a left and a right ideal, than A is a two sided ideal or simply an ideal of M.

Definition 2.5[11]. A fuzzy set A in M is a function $A : M \to [0, 1]$.

Definition 2.6[11]. Let μ be a fuzzy set in a Γ - ring M. For any $t \in [0, 1]$, the set $U(\mu, t) = \{x \in M | \mu(x) \geq t\}$ is called a level set of μ.

Definition 2.7[11]. A fuzzy set μ in a Γ - ring M is called a fuzzy left (resp. right) ideal of M, if it satisfies:

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(x\alpha y) \geq \mu(y)$ (resp. $\mu(x\alpha y) \geq \mu(x)$),

for all $x, y \in M$ and $\alpha \in \Gamma$. If μ is both a fuzzy left and right ideal of M, then μ is called a fuzzy ideal of M.

Definition 2.8[1]. Let X be a nonempty fixed set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$, for every $x \in X$.

Notation. For the sake of simplicity, we shall denote the intuitionistic fuzzy set (IFS in short) $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ by $A = \langle \mu_A, \nu_A \rangle$.

Definition 2.9[1]. Let X be a non empty set and let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be IFSs in X. Then

1. $A \subset B$ iff $\mu_A \le \mu_B$ and $\nu_A \ge \nu_B$.

2. $A = B$ iff $A \subset B$ and $B \subset A$.

3. $A^c = \langle \nu_A, \mu_A \rangle$.

4. $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

5. $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

6. $\Box A = (\mu_A, 1 - \mu_A)$, $\Diamond A = (1 - \nu_A, \nu_A)$.

Definition 2.10[5]. Let A be an IFS in a Γ - ring M. For each pair $\langle t, s \rangle \in [0, 1]$ with $t + s \le 1$, the set $A_{\langle t, s \rangle} = \{x \in X \mid \mu_A(x) \ge t \text{ and } \nu_A(x) \le s\}$ is called a $\langle t, s \rangle$ - level subset of A.

Definition 2.11.[6]. Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy set in a Γ - ring M and let $t \in [0, 1]$. Then the sets $U(\mu_A; t) = \{x \in M : \mu_A(x) \ge t\}$ and $L(\nu_A; t) = \{x \in M : \nu_A(x) \le t\}$ are called upper level set and lower level set of A respectively.

3 Intuitionistic fuzzy ideals

In what follows, let M denote a Γ - ring unless otherwise specified. In this section, an example of an intuitionistic fuzzy ideal is given.

Definition 3.1. An IFS $A = \langle \mu_A, \nu_A \rangle$ in M is called an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M if

(i) $\mu_A(x - y) \ge \{\mu_A(x) \wedge \mu_A(y)\}$ and $\mu_A(x\alpha y) \ge \mu_A(y)$ $\big(\text{resp. } \mu_A(x\alpha y) \ge \mu_A(x)\big)$,

(ii) $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$ and $\nu_A(x\alpha y) \leq \nu_A(y)\big(\text{resp.}\nu_A(x\alpha y) \leq \nu_A(x)\big)$,

for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 3.2. [Intuitionistic fuzzy ideal of a Γ-ring]
Let R be the set of all integers. Then R is a ring.
Take M = Γ = R.
Let $a, b \in M, \alpha \in \Gamma$. Suppose $a\alpha b$ is the product of $a, \alpha, b \in M$. Then M is a Γ-ring.
Define an IFS A $= \langle \mu_A, \nu_A \rangle$ in M as follows.
$\mu_A(0) = 1$ and $\mu_A(\pm 1) = \mu_A(\pm 2) = \ldots .= t$ and
$\nu_A(0) = 0$ and $\nu_A(\pm 1) = \nu_A(\pm 2) = \ldots .= s$, where $t \in [0, 1], s \in [0, 1]$ *and* $t + s \leq 1$.
By routine calculations, clearly A is an intuitionistic fuzzy ideal of a Γ-ring M.

Theorem 3.3. If A is an ideal of a Γ - ring M, then the IFS $\hat{A} = \langle \chi_A, \overline{\chi_A} \rangle$ is an intuitionistic fuzzy ideal of M.
Proof. Let $x, y \in M$.
If $x, y \in A$ and $\alpha \in \Gamma$, then $x - y \in A$ and $x\alpha y \in A$, since A is an ideal of M.
Hence $\chi_A(x - y) = 1 \geq \{\chi_A(x) \wedge \chi_A(y)\}$ and $\chi_A(x\alpha y) = 1 \geq \chi_A(y)$.
Also, we have
$0 = 1 - \chi_A(x - y) = \overline{\chi_A}(x - y) \leq \{\overline{\chi_A}(x) \vee \overline{\chi_A}(y)\}$ and
$0 = 1 - \chi_A(x\alpha y) = \overline{\chi_A}(x\alpha y) \leq \overline{\chi_A}(y)$.
If $x \notin A$ or $y \notin A$, then $\chi_A(x) = 0$ or $\chi_A(y) = 0$. Thus wwe have
$\chi_A(x - y) \geq \{\chi_A(x) \wedge \chi_A(y)\}$ and
$chi_A(x\alpha y) \geq \chi_A(y)$ for all $\alpha \in \Gamma$.
Also $\overline{\chi_A}(x - y) \leq \{\overline{\chi_A}(x) \vee$
$overline\chi_A(y)\} = \big(1 - \chi_A(x)\big) \vee \big(1 - \chi_A(y)\big) = 1$
and $\overline{\chi_A}(x\alpha y) = 1 - \chi_A(x\alpha y) \leq 1 - \chi_A(y) = \overline{\chi_A}(y)$.
This completes the proof.

Definition 3.4[3]. An intuitionistic fuzzy left (resp. right) ideal $A = \langle \mu_A, \nu_A \rangle$ of a Γ - ring M is said to be normal if $\mu_A(0) = 1$ and $\nu_A(0) = 0$.

Theorem 3.5. Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy left (resp. right) ideal of a Γ-ring M and let $\mu_A^+(x) = \mu_A(x) + 1 - \mu_A(0), \nu_A^+(x) = \nu_A(x) - \nu_A(0)$. If $\mu_A^+(x) + \nu_A^+(x) \leq 1$ for all $x \in M$, then $A^+ = \langle \mu_A^+, \nu_A^+ \rangle$ is a normal intuitionistic fuzzy left (resp. right) ideal of M.
Proof. We first observe that $\mu_A^+(0) = 1, \nu_A^+(0) = 0$ and $\mu_A^+(x), \nu_A^+(x) \in [0, 1]$ for every $x \in M$. So, $A^+ = \langle \mu_A^+, \nu_A^+ \rangle$ is a normal intuitionistic fuzzy set.
To prove that it is an intuitionistic fuzzy left (resp. right) ideal, let $x, y \in M$ and $\alpha \in \Gamma$.

Then $\mu_A^+(x - y) = \mu_A(x - y) + 1 - \mu_A(0) \geq \{\mu_A(x) \wedge \mu_A(y)\} + 1 - \mu_A(0)$

$$= \{\mu_A(x) + 1 - \mu_A(0)\} \wedge \{\mu_A(y) + 1 - \mu_A(0)\} = \mu_A^+(x) \wedge \mu_A^+(y),$$

$$\nu_A^+(x - y) = \nu_A(x - y) - \nu_A(0) \leq \{\nu_A(x) \vee \nu_A(y)\} - \nu_A(0)$$

$$= \{\nu_A(x) - \nu_A(0)\} \vee \{\nu_A(y) - \nu_A(0)\} = \nu_A^+(x) \vee \nu_A^+(y),$$

and $\mu_A^+(x\alpha y) = \mu_A(x\alpha y) + 1 - \mu_A(0) \geq \mu_A(y) + 1 - \mu_A(0) = \mu_A^+(y),$

$$\nu_A^+(x\alpha y) = \nu_A(x\alpha y) - \nu_A(0) \leq \nu_A(y) - \nu_A(0) = \nu_A^+(y).$$

This shows that A^+ is an intuitionistic fuzzy left(resp. right) ideal of M.
So, $A^+ = \langle \mu_A^+, \nu_A^+ \rangle$ is a normal intuitionistic fuzzy left(resp. right)ideal of M.

Definition 3.6 [2]. Let I be an ideal of a Γ - ring M. If for each $a + I$, $b + I$ in the factor group M/I and each $\alpha \in \Gamma$, we define $(a + I)\alpha(b + I) = a\alpha b + I$, then M/I is a Γ - ring which we shall call the Γ - residue class ring of M with respect to I.

Theorem 3.7. Let I be an ideal of a Γ - ring M. If A is an intuitionistic fuzzy left (resp. right) ideal of M, then the intuitionistic fuzzy set $\tilde{A}$ of M/I defined by

$$\mu_{\tilde{A}}(a + I) = \bigvee_{x \in I} \mu_A(a + x) \text{ and } \nu_{\tilde{A}}(a + I) = \bigwedge_{x \in I} \nu_A(a + x)$$

is an intuitionistic fuzzy left (resp. right) ideal of the Γ - residue class ring M/I of M with respect to I.
Proof. Let $a, b \in M$ be such that $a + I = b + I$.
Then $b = a + y$ for some $y \in I$ and so

$$\mu_{\tilde{A}}(b+I) = \bigvee_{x \in I} \mu_A(b+x) = \bigvee_{x \in I} \mu_A(a+y+x) = \bigvee_{x+y=z \in I} \mu_A(a+z) = \mu_{\tilde{A}}(a+I),$$

$$\nu_{\tilde{A}}(b+I) = \bigwedge_{x \in I} \nu_A(b+x) = \bigwedge_{x \in I} \nu_A(a+y+x) = \bigwedge_{x+y=z \in I} \nu_A(a+z) = \nu_{\tilde{A}}(a+I).$$

Hence $\tilde{A}$ is well defined.
For any $x + I, y + I \in M/I$ and $\alpha \in \Gamma$, we have

$$\mu_{\tilde{A}}\big((x + I) - (y + I)\big) = \mu_{\tilde{A}}\big((x - y) + I\big) = \bigvee_{z \in I} \mu_A\big((x - y) + z\big)$$

$$= \bigvee_{z=u-v \in I} \mu_A\big((x - y) + (u - v)\big)$$

$$= \bigvee_{u,v \in I} \mu_A\big((x + u) - (y + v)\big)$$

$$\geq \bigvee_{u,v \in I} \big(\mu_A(x + u) \wedge \mu_A(y + v)\big)$$

$$= \big(\bigvee_{u \in I} \mu_A(x+u)\big) \wedge \big(\bigvee_{v \in I} \mu_A(y+v)\big)$$

$$= \mu_{\tilde{A}}(x + I) \wedge \mu_{\tilde{A}}(y + I),$$

$$\nu_{\tilde{A}}\big((x + I) - (y + I)\big) = \nu_{\tilde{A}}\big((x - y) + I\big) = \bigwedge_{z \in I} \nu_A\big((x - y) + z\big)$$

$$= \bigwedge_{z=u-v\in I} \nu_A\big((x-y)+(u-v)\big)$$

$$= \bigwedge_{u,v\in I} \nu_A\big((x+u)-(y+v)\big)$$

$$\leq \bigwedge_{u,v\in I} \big(\nu_A(x+u)\vee\nu_A(y+v)\big)$$

$$= \Big(\bigwedge_{u\in I}\nu_A(x+u)\Big)\vee\Big(\bigwedge_{v\in I}\nu_A(y+v)\Big)$$

$$= \nu_{\tilde{A}}(x+I)\vee\nu_{\tilde{A}}(y+I),$$

$$\mu_{\tilde{A}}\big((x+I)\alpha(y+I)\big) = \mu_{\tilde{A}}\big((x\alpha y)+I\big) = \bigvee_{z\in I}\mu_A\big((x\alpha y)+z\big)$$

$$\geq \bigvee_{z\in I}\mu_A\big(x\alpha y+x\alpha z\big)\ \text{because } x\alpha z\in I$$

$$= \bigvee_{z\in I}\mu_A\big(x\alpha(y+z)\big)$$

$$\geq \bigvee_{z\in I}\mu_A(y+z)$$

$$= \mu_{\tilde{A}}(y+I),$$

$$\nu_{\tilde{A}}\big((x+I)\alpha(y+I)\big) = \nu_{\tilde{A}}\big((x\alpha y)+I\big) = \bigwedge_{z\in I}\nu_A\big((x\alpha y)+z\big)$$

$$\leq \bigwedge_{z\in I}\nu_A\big(x\alpha y+x\alpha z\big)\ \text{because } x\alpha z\in I$$

$$= \bigwedge_{z\in I}\nu_A\big(x\alpha(y+z)\big)$$

$$\leq \bigwedge_{z\in I}\nu_A(y+z)$$

$$= \nu_{\tilde{A}}(y+I).$$

Similarly,

$$\mu_{\tilde{A}}\big((x+I)\alpha(y+I)\big) \geq \mu_{\tilde{A}}(x+I) \text{ and } \nu_{\tilde{A}}\big((x+I)\alpha(y+I)\big) \leq \nu_{\tilde{A}}(x+I).$$

Hence $\tilde{A}$ is an intuitionistic fuzzy left (resp. right) ideal of M/I.

Theorem 3.8. If the IFS $A = \langle\mu_A,\nu_A\rangle$ is an intuitionistic fuzzy left (resp. right) ideal of M, then the set $M_A = \{x\in M|\mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$ is an ideal of M.

Proof. Let $x,y\in M_A$.

Then $\mu_A(x) = \mu_A(y) = \mu_A(0)$ and $\nu_A(x) = \nu_A(y) = \nu_A(0)$.

Since A is an intuitionistic fuzzy ideal of M, it follows that

$\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\} = \{\mu_A(0) \wedge \mu_A(0)\} = \mu_A(0)$,

$\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} = \{\nu_A(0) \vee \nu_A(0)\} = \nu_A(0)$.

Hence $\mu_A(x - y) = \mu_A(0)$ and $\nu_A(x - y) = \nu_A(0)$. So $x - y \in M_A$.

Let $x \in M, \alpha \in \Gamma$ and $y \in M_A$.

Therefore $\mu_A(x\alpha y) \geq \mu_A(y) = \mu_A(0)$ (resp. $\mu_A(x\alpha y) \geq \mu_A(x) = \mu_A(0)$) and

$\nu_A(x\alpha y) \leq \nu_A(y) = \nu_A(0)$ (resp. $\nu_A(x\alpha y) \leq \nu_A(x) = \nu_A(0)$).

Hence $\mu_A(x\alpha y) = \mu_A(0)$ and $\nu_A(x\alpha y) = \nu_A(0)$.
So xαy $\in M_A$.Hence M_A is an ideal of M.

Theorem 3.9. Let A be an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M. For each pair $\langle t, s \rangle \in [0, 1]$, the level set $A_{\langle t,s \rangle}$ is an ideal of M.

Proof. Let $x, y \in A_{\langle t,s \rangle}$.
Then $\mu_A(x) \geq t, \mu_A(y) \geq t$ and $\nu_A(x) \leq s, \nu_A(y) \leq s$.
Since A is an intuitionistic fuzzy left (resp. right) ideal, we have
$\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\} \geq t$ and $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} \leq s$.
So $x - y \in A_{\langle t,s \rangle}$.
Let $x \in M, y \in A_{\langle t,s \rangle}$ and $\alpha \in \Gamma$.
Then $\mu_A(x\alpha y) \geq \mu_A(y) \geq t$ and $\nu_A(x\alpha y) \leq \nu_A(y) \leq s$. So $x\alpha y \in A_{\langle t,s \rangle}$.
Hence $A_{\langle t,s \rangle}$ is an ideal of M.

Definition 3.10. Let A and B be two intuitionistic fuzzy subsets of a Γ - ring M and $\alpha \in \Gamma$. Then the product $A\Gamma B$ is defined by
$$\mu_{A\Gamma B}(x) = \bigvee_{x=y\alpha z} \big(\mu_A(y) \wedge \mu_A(z) \big) \text{ if } x = y\alpha z,$$

$$\nu_{A\Gamma B}(x) = \bigwedge_{x=y\alpha z} \big(\nu_A(y) \vee \nu_A(z) \big) \text{ if } x = y\alpha z.$$

Otherwise, we define $\mu_{A\Gamma B}(x) = 0$ and $\nu_{A\Gamma B}(x) = 1$.

Definition 3.11[4]. Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be two IFSs in a Γ - ring M. Then the composition of A and B is defined to be the intuitionistic fuzzy set $A \circ B = \langle \mu_{A \circ B}, \nu_{A \circ B} \rangle$ in M given by

$$\mu_{A \circ B}(x) = \bigvee \Big\{ \bigwedge_{1 \leq i \leq k} \mu_A(a_i) \wedge \mu_B(b_i) : x = \sum_1^k a_i \alpha b_i, a_i, b_i \in M, \alpha \in \Gamma, k \in N$$

$$\nu_{A \circ B}(x) = \bigwedge \Big\{ \bigvee_{1 \leq i \leq k} \nu_A(a_i) \vee \nu_B(b_i) : x = \sum_1^k a_i \alpha b_i, a_i, b_i \in M, \alpha \in \Gamma, k \in N$$

if we can express $x = \sum_{i=1}^k a_i \alpha b_i$ for some $a_i, b_i \in M$, where each $a_i \alpha b_i \neq 0$ and $k \in N$.

Otherwise, we define $A \circ B = 0$, i.e., $\mu_{A \circ B}(x) = 0$ and $\nu_{A \circ B}(x) = 1$.

Theorem 3.12. If $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ are intuitionistic fuzzy ideals in a Γ - ring M then $A \circ B$ is an intuitionistic fuzzy ideal in M.
Proof. For any $x, y \in M$, we have

$\mu_{A \circ B}(x - y)$

$= \bigvee \left\{ \bigwedge_{1 \le i \le k} \mu_A(u_i) \wedge \mu_B(v_i) : x - y = \sum_1^k u_i \alpha v_i, u_i, v_i \in M, \alpha \in \Gamma \text{ and } k \in N \right\}$

$\ge \bigvee \left\{ \left(\bigwedge_{1 \le i \le m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \left(\bigwedge_{1 \le i \le n} \mu_A(-c_i) \wedge \mu_B(d_i) \right) \right.$

$\left. : x = \sum_1^m a_i \alpha b_i, -y = \sum_1^n -c_i \alpha d_i, a_i, b_i, -c_i, d_i \in M, \alpha \in \Gamma \text{ and } m, n \in N \right\}$

$= \bigvee \left\{ \left(\bigwedge_{1 \le i \le m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \left(\bigwedge_{1 \le i \le n} \mu_A(c_i) \wedge \mu_B(d_i) \right) \right.$

$\left. : x = \sum_1^m a_i \alpha b_i, y = \sum_1^n c_i \alpha d_i, a_i, b_i, c_i, d_i \in M, \alpha \in \Gamma \text{ and } m, n \in N \right\}$

$= \bigvee \left\{ \bigwedge_{1 \le i \le m} \mu_A(a_i) \wedge \mu_B(b_i) : x = \sum_1^m a_i \alpha b_i, a_i, b_i, \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$

$\qquad \wedge \bigvee \left\{ \bigwedge_{1 \le i \le n} \mu_A(c_i) \wedge \mu_B(d_i) : y = \sum_1^n c_i \alpha d_i, c_i, d_i, \in M, \alpha \in \Gamma \text{ and } n \in N \right\}$

$= \mu_{A \circ B}(x) \wedge \mu_{A \circ B}(y)$

$\nu_{A \circ B}(x - y)$

$= \bigwedge \left\{ \bigvee_{1 \le i \le k} \nu_A(u_i) \vee \nu_B(v_i) : x - y = \sum_1^k u_i \alpha v_i, u_i, v_i \in M, \alpha \in \Gamma \text{ and } k \in N \right\}$

$\le \bigwedge \left\{ \left(\bigvee_{1 \le i \le m} \nu_A(a_i) \vee \nu_B(b_i) \right) \vee \left(\bigvee_{1 \le i \le n} \nu_A(-c_i) \vee \nu_B(d_i) \right) \right.$

$\left. : x = \sum_1^m a_i \alpha b_i, -y = \sum_1^n -c_i \alpha d_i, a_i, b_i, -c_i, d_i \in M, \alpha \in \Gamma \text{ and } m, n \in N \right\}$

$= \bigwedge \left\{ \left(\bigvee_{1 \le i \le m} \nu_A(a_i) \vee \nu_B(b_i) \right) \vee \left(\bigvee_{1 \le i \le n} \nu_A(c_i) \vee \nu_B(d_i) \right) \right.$

$\left. : x = \sum_1^m a_i \alpha b_i, y = \sum_1^n c_i \alpha d_i, a_i, b_i, c_i, d_i \in M, \alpha \in \Gamma \text{ and } m, n \in N \right\}$

$= \bigwedge \left\{ \bigvee_{1 \le i \le m} \nu_A(a_i) \vee \nu_B(b_i) : x = \sum_1^m a_i \alpha b_i, a_i, b_i \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$

$\qquad \vee \bigwedge \left\{ \bigvee_{1 \le i \le n} \nu_A(c_i) \vee \nu_B(d_i) : y = \sum_1^n c_i \alpha d_i, c_i, d_i \in M, \alpha \in \Gamma \text{ and } n \in N \right\}$

$= \nu_{A \circ B}(x) \vee \nu_{A \circ B}(y).$

Also,we have

$$\mu_{A \circ B}(x) = \bigvee \left\{ \bigwedge_{1 \le i \le m} \mu_A(a_i) \wedge \mu_B(b_i) : x = \sum_1^m a_i \alpha b_i, a_i, b_i \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$\le \bigvee \left\{ \bigwedge_{1 \le i \le m} \mu_A(a_i) \wedge \mu_B(b_i \alpha y) : x \alpha y = \sum_1^m a_i \alpha(b_i \alpha y), a_i, b_i \alpha y \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$\le \bigvee \left\{ \bigwedge_{1 \le i \le m} \mu_A(u_i) \wedge \mu_B(v_i) : x \alpha y = \sum_1^m u_i \alpha v_i, u_i, v_i \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$= \mu_{A \circ B}(x \alpha y)$$

$$\nu_{A \circ B}(x) = \bigwedge \left\{ \bigvee_{1 \le i \le m} \nu_A(a_i) \vee \nu_B(b_i) : x = \sum_1^m a_i \alpha b_i, a_i, b_i \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$\ge \bigwedge \left\{ \bigvee_{1 \le i \le m} \nu_A(a_i) \vee \nu_B(b_i \alpha y) : x \alpha y = \sum_1^m a_i \alpha(b_i \alpha y), a_i, b_i \alpha y \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$\ge \bigwedge \left\{ \bigvee_{1 \le i \le m} \nu_A(u_i) \vee \nu_B(v_i) : x \alpha y = \sum_1^m u_i \alpha v_i, u_i, v_i \in M, \alpha \in \Gamma \text{ and } m \in N \right\}$$

$$= \nu_{A \circ B}(x \alpha y).$$

Hence $\mu_{A \circ B}(x \alpha y) \ge \mu_{A \circ B}(x)$ and $\nu_{A \circ B}(x \alpha y) \le \nu_{A \circ B}(x)$.

Similarly we get $\mu_{A \circ B}(x \alpha y) \ge \mu_{A \circ B}(y)$ and $\nu_{A \circ B}(x \alpha y) \le \nu_{A \circ B}(y)$.

Therefore $A \circ B = (\mu_{A \circ B}, \nu_{A \circ B})$ is an intuitionistic fuzzy ideal of M.

Definition 3.13[2]. A function $f : M \to N$, where M and N are Γ - rings, is said to be a Γ - homomorphism if $f(a + b) = f(a) + f(b)$, $f(a \alpha b) = f(a) \alpha f(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 3.14[2]. A function $f : M \to N$, where f is a Γ - homomorphism and M and N are Γ - rings, is said to be a Γ - endomorphism if $N \subseteq M$.

Definition 3.15[2]. Let $f : X \to Y$ be a mapping of Γ - rings and A be an intuitionistic fuzzy set of Y. Then the map $f^{-1}(A)$ is the pre-image of A under f, if $\mu_{f^{-1}(A)}(X) = \mu_A(f(x))$ and $\nu_{f^{-1}(A)}(X) = \nu_A(f(x))$, for all $x \in X$.

Theorem 3.16 Let f be a Γ - homomorphism of M. If the IFS $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy left (resp. right) ideal of M, then $B = \langle \mu_{f^{-1}(A)}, \nu_{f^{-1}(A)} \rangle$ is an intuitionistic fuzzy left (resp. right) ideal of M.

Proof. For any $x, y \in M$, $\alpha \in \Gamma$, we have

$$\mu_{f^{-1}(A)}(x - y) = \mu_A\big(f(x - y)\big)$$
$$= \mu_A\big(f(x) - f(y)\big) \ge \mu_A\big(f(x)\big) \wedge \mu_A\big(f(y)\big) = \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)$$
and $\mu_{f^{-1}(A)}(x \alpha y) = \mu_A\big(f(x \alpha y)\big) = \mu_A\big(f(x) \alpha f(y)\big) \ge \mu_{f^{-1}(A)}(y)$.
Similarly, $\nu_{f^{-1}(A)}(x - y) = \nu_A\big(f(x - y)\big)$
$$= \nu_A\big(f(x) - f(y)\big) \le \nu_A\big(f(x)\big) \vee \nu_A\big(f(y)\big) = \nu_{f^{-1}(A)}(x) \vee \nu_{f^{-1}(A)}(y)$$
and $\nu_{f^{-1}(A)}(x \alpha y) = \nu_A\big(f(x \alpha y)\big) = \nu_A\big(f(x) \alpha f(y)\big) \le \nu_{f^{-1}(A)}(y)$.

Hence B is an intuitionistic fuzzy left (resp. right) ideal of M.

Theorem 3.17. If $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy set in M such that the

non-empty sets $U(\mu_A; t)$ and $L(\nu_A; t)$ are ideals of M for all $t \in [0, 1]$, then A is an intuitionistic fuzzy left (resp. right) ideal of M.

Proof. Suppose that there exists $x_0, y_0 \in M$ such that

$\mu_A(x_0 - y_0) < \big(\mu_A(x_0) \wedge \mu_A(y_0)\big)$.

Let $t_0 = \frac{1}{2}\big\{\mu_A(x_0 - y_0) + \big(\mu_A(x_0) \wedge \mu_A(y_0)\big)\big\}$.

Then $\big(\mu_A(x_0) \wedge \mu_A(y_0)\big) \geq t_0 > \mu_A(x_0 - y_0)$.

It follows that $x_o, y_o \in U(\mu_A; t_0)$ and $x_0 - y_0 \notin U(\mu_A; t_0)$.

This is a contradiction.

Hence $\mu_A(x - y) \geq \big(\mu_A(x) \wedge \mu_A(y)\big)$, for all $x, y \in M$.

Now let $x_0, y_0 \in M$ and $\alpha \in \Gamma$ such that $\mu_A(x_0 \alpha y_0) < \mu_A(y_0)$.

Let $t_0 = \frac{1}{2}\big\{\mu_A(x_0 \alpha y_0) + \mu_A(y_0)\big\}$.

Then we get $\mu_A(x_0 \alpha y_0) \leq t_0 < \mu_A(y_0)$.

It follows that $y_0 \in U(\mu_A; t_0)$ and $x_0 \alpha y_0 \notin U(\mu_A; t_0)$.

This is a contradiction.

Thus $\mu_A(x_0 \alpha y_0) \geq \mu_A(y_0)$ (resp. $\mu_A(x_0 \alpha y_0) \geq \mu_A(x_0)$).

Similarly, suppose that there exists $x_0, y_0 \in M$ such that

$\nu_A(x_0 - y_0) > \big\{\nu_A(x_0) \vee \nu_A(y_0)\big\}$.

Let $t_0 = \frac{1}{2}\big\{\nu_A(x_0 - y_0) + \big(\nu_A(x_0) \vee \nu_A(y_0)\big)\big\}$.

Then $\big(\nu_A(x_0) \vee \nu_A(y_0)\big) \leq t_0 < \nu_A(x_0 - y_0)$.

It follows that $x_o, y_o \in L(\mu_A; t_0)$ and $x_0 - y_0 \notin L(\mu_A; t_0)$.

This is a contradiction.

Hence $\nu_A(x - y) \leq \big(\nu_A(x) \vee \nu_A(y)\big)$, for all $x, y \in M$.

Now let $x_0, y_0 \in M$ and $\alpha \in \Gamma$ such that $\nu_A(x_0 \alpha y_0) > \nu_A(y_0)$.

Let $t_0 = \frac{1}{2}\big\{\nu_A(x_0 \alpha y_0) + \nu_A(y_0)\big\}$.

Then we get $\nu_A(x_0 \alpha y_0) > t_0 > \nu_A(y_0)$.

It follows that $y_0 \in L(\mu_A; t_0)$ and $x_0 \alpha y_0 \notin L(\nu_A; t_0)$.

This is a contradiction.

Thus $\nu_A(x_0 \alpha y_0) \leq \nu_A(y_0)$ (resp. $\nu_A(x_0 \alpha y_0) \leq \nu_A(x_0)$).

Hence A is an intuitionistic fuzzy left (resp. right) ideal of M.

Acknowledgement

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A new class of analytic functions based on Ruscheweyh derivative

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ABSTRACT: In this paper we introduce a new class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ consisting of analytic functions with negative coefficients and investigate various properties and characterization of the class. The results include coefficient estimates, distortion theorem, closure theorems and integral operators for the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$. Also radii of close-to-convexity, starlikeness and convexity are determined

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of functions univalent in Δ.

Let f and g be functions analytic in Δ. Then we say that f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that
$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$
We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta. \tag{1.2}$$

By using the Hadamard product, Ruscheweyh [10] defined an operator

$$D^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), \quad \gamma \geq -1. \tag{1.3}$$

Ruscheweyh [10] observed that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \tag{1.4}$$

where $n = \gamma \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. This symbol $D^n f(z)$, $n \in \mathbb{N}_0$ is called by Al-Amiri [1], the nth order Ruscheweyh derivative of $f(z)$.

We note that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, and

$$D^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n, k) a_k z^k, \tag{1.5}$$

where

$$\sigma(n, k) = \binom{n+k-1}{n} \tag{1.6}$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1.7}$$

Several new classes of analytic functions defined by Ruscheweyh derivatives have been studied and continue to be introduced and investigated in the literature (See for example [12, 13, 8] to mention a few interesting studies). Several investigations on functions with negative coefficients have been done. (See for example [3, 2, 6, 9, 11]). In particular, results on functions with negative coefficients related to Ruscheweyh derivatives have been derived. See for example, [4, 5, 7]).

Motivated by the aforementioned works, we introduce a new class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ by using m^{th} and n^{th} order Ruscheweyh derivative of $f(z)$.

Definition 1 *Let* $\Phi(z) = z + \sum_{k=2}^{\infty} \beta_k z^k$, *and* $\Psi(z) = z + \sum_{k=2}^{\infty} \gamma_k z^k$ *be fixed analytic functions in* Δ *and* $\beta_k > 0$, $\gamma_k > 0$, $k \geq 2$. *We define a class* $Q_m^n(\Phi, \Psi, \lambda, A, B)$

consisting of analytic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0,$$

which satisfy the subordination condition

$$(1-\lambda)(D^m(f * \Phi)(z))' + \lambda(D^n(f * \Psi)(z))' \prec \frac{1+Az}{1+Bz} \tag{1.8}$$

for $z \in \Delta$, where $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $m, n \in N_0$.

By specializing the parameters m, n, λ, A and B, and the functions Φ and Ψ, we obtain the subclasses studied by various authors as listed below:

(i) $Q_m^{m+1}\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, \lambda, A, B\right) = Q(m, \lambda, A, B)$ [4]

(ii) $Q_0^1\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, \lambda, 2\alpha - 1, 1\right) = R(\lambda, \alpha)$ $(0 \leq \alpha < 1)$ [3]

(iii) $Q_0^1\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, 0, 2\alpha - 1, 1\right) = T^*(\alpha)$ $(0 \leq \alpha < 1)$ [11, 2]

(iv) $Q_m^{m+1}\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, 0, 2\alpha - 1, 1\right) = Q_n(\alpha)$ $(0 \leq \alpha < 1)$ [14]

(v) $Q_0^1\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, 0, (2\alpha - 1)\beta, \beta\right) = p^*(\alpha, \beta)$
  $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ [6]

(vi) $Q_0^1\left(\dfrac{z}{1-z}, \dfrac{z}{1-z}, 0, ((1+\mu)\alpha - 1)\beta, \mu\beta\right) = p^*(\alpha, \beta, \mu)$
  $(0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \mu < 1)$ [9]

It is of interest to note that the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ gives several well-known subclasses of functions for suitable choices of $\Phi(z)$, $\Psi(z)$ and m, n. We obtain coefficient inequality, coefficient estimate, distortion theorem, extreme points and integral representation for functions in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$.

2   Coefficient Estimates

Theorem 1 *A function $f(z) = z - \displaystyle\sum_{k=2}^{\infty} a_k z^k \in Q_m^n(\Phi, \Psi, \lambda, A, B)$ if and only if*

$$\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_k \leq \frac{B-A}{B+1} \tag{2.1}$$

Proof.   Suppose $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$. In view of the definition of subordination we get,

$$h(z) = (1 - \lambda)(D^m(f * \Phi)(z))' + \lambda(D^n(f * \Psi)(z))' = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad (2.2)$$

$-1 \leq A < B \leq 1$, $0 < B \leq 1$, $z \in \Delta$, and $|\omega(z)| < 1$. From (2.2), we get

$$w(z) = \frac{1 - h(z)}{Bh(z) - A}.$$

In view of (2.2), one can easily obtain through a simple computation that,

$$h(z) = 1 - \sum_{k=0}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k z^{k-1}$$

and $|w(z)| < 1$ implies

$$\left| \frac{\displaystyle\sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k z^{k-1}}{(B - A) - B\displaystyle\sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k z^{k-1}} \right| < 1 \qquad (2.3)$$

and hence,(2.1)holds.

Conversely, Suppose $f \in T$ and satisfies (2.1). For $|z| = r$, $0 \leq r < 1$, we have by (2.1),

$$|1 - h(z)| - |Bh(z) - A| \leq \sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k r^{k-1}$$

$$-(B - A) + B\sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k r^{k-1} \leq 0$$

which gives (2.2) and hence follows that

$$(1 - \lambda)(D^m(f * \Phi)(z))' + \lambda(D^n(f * \Psi)(z))' = \frac{1 + Aw(z)}{1 + Bw(z)},$$

$z \in \Delta$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Hence, $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$.

Finally the function $f(z)$ given by

$$f(z) = z - \frac{B - A}{k(1 + B)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]} z^k, \quad k \geq 2 \qquad (2.4)$$

is an extremal function for the class. ∎

Corollary 1 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$. Then we have*

$$a_n \leq \frac{B - A}{k(1 + B)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]}, \quad k \geq 2. \tag{2.5}$$

Theorem 2
$$Q_m^n(\Phi, \Psi, \lambda_2, A, B) \subseteq Q_m^n(\Phi, \Psi, \lambda_1, A, B)$$

for $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $\lambda_2 \geq \lambda_1 \geq 0$ and $n > m$.

Proof. Let $f(z) \in Q_m^n(\Phi, \Psi, \lambda_2, A, B)$.
$$\sum_{k=2}^{\infty} k\left[(1 - \lambda_1)\sigma(m, k)\beta_k + \lambda_1\sigma(n, k)\gamma_k\right]a_k$$

$$\leq \sum_{k=2}^{\infty} k\left[\sigma(m, k)\beta_k + (\sigma(n, k)\gamma_k - \sigma(m, k)\beta_k)\lambda_2\right]a_k \leq \frac{B - A}{1 + B}$$

Therefore $f(z) \in Q_m^n(\Phi, \Psi, \lambda_1, A, B)$. Hence the proof of Theorem 2 is complete. ∎

3 Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \ldots, \ell$ by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,\ell}z^k, \quad a_{k,\ell} \geq 0. \tag{3.1}$$

We shall prove the following results for the closure of functions in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$.

Theorem 3 *Let the function $f_i(z)$ defined by (3.1) be in the class $Q_m^n(\Phi, \Psi, \lambda, A_i, B_i)$, for $i = 1, 2, \ldots, \ell$. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{\ell}\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\ell} a_{k,i}z^k\right) \tag{3.2}$$

is in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$,

$$\text{where} \quad A = \min_{1 \leq i \leq \ell}\{A_i\} \quad \text{and} \quad B = \max_{1 \leq i \leq \ell}\{B_i\}. \tag{3.3}$$

Proof. Since $f_i(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$, for $i = 1, 2, \ldots, \ell$, we have

$$\sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]a_{k,i} \leq \frac{B_i - A_i}{1 + B_i}, \text{ by Theorem 1}$$

Hence we obtain,

$$\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] \left[\frac{1}{\ell}\sum_{i=1}^{\ell} a_{k,i}\right]$$

$$\leq \frac{1}{\ell}\sum_{i=1}^{\ell}\left\{\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_{k,i}\right\}$$

$$\leq \frac{1}{\ell}\sum_{i=1}^{\ell} \frac{B_i - A_i}{1 + B_i} \leq \frac{B-A}{1+B}.$$

Thus, we get

$$\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] \left[\frac{1}{\ell}\sum_{i=1}^{\ell} a_{k,i}\right] \leq \frac{B-A}{1+B}$$

given (3.3), which shows that $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$. ∎

Theorem 4 *Let the functions $f_i(z)$ $(i = 1, 2, \ldots, \ell)$ defined by (3.1) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$. Then the function $h(z)$ defined by $h(z) = \sum_{i=1}^{\ell} d_i f_i(z)$ is also in the same class $Q_m^n(\Phi, \Psi, \lambda, A, B)$, where $\sum_{i=1}^{\ell} d_i = 1$.*

Proof.   According to definition of $h(z)$ we can write that

$$h(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\ell} d_i a_{k,i}\right) z^k.$$

Further, since $f_i(z)$ are in $Q_m^n(\Phi, \Psi, \lambda, A, B)$ for every $i = 1, 2, \ldots, \ell$, we get

$$\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_{k,i} \leq \frac{B-A}{1+B}$$

for every $i = 1, 2, \ldots, \ell$.

Hence, we can see that

$$\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] \left(\sum_{i=1}^{\ell} d_i a_{k,i}\right)$$

$$= \sum_{i=1}^{\ell} d_i \left(\sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_{k,i}\right)$$

$$\leq \left(\sum_{i=1}^{\ell} d_i\right) \frac{B-A}{1+B} = \frac{B-A}{1+B}.$$

This proves that the function $h(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$. Thus we have the theorem.
∎

Corollary 2 *The class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ is closed under convex linear combination.*

Proof. Putting $\ell = 2$ in the above theorem, we prove the corollary. ■

Theorem 5 *Let*
$$f_1(z) = z$$
and
$$f_k(z) = z - \frac{B - A}{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]}z^k, \ k \geq 2.$$
Then $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$ if and only if it can be expressed in the form
$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{3.4}$$
where $\mu_k \geq 0 \ (k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Assume that $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$. Then

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(B - A)\mu_k}{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]}z^k \tag{3.5}$$

Then it follows that

$$\sum_{k=2}^{\infty} \frac{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]}{B - A}$$

$$\cdot \frac{(B - A)\mu_k}{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]} = \sum_{k=2}^{\infty} \mu_k$$
$$= 1 - \mu_1$$
$$\leq 1$$

Then $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$.

Conversely, assume that $f(z)$ defined by (1.7) belongs to the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$. Then

$$a_k \leq \frac{(B - A)}{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]} \quad (k \geq 2).$$

Setting,

$$\mu_k = \frac{k(B + 1)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right]}{(B - A)}, \quad \mu_1 \quad = 1 - \sum_{k=2}^{\infty} \mu_k$$

we can see that $f(z)$ can be expressed in the form (3.4). This completes the proof. ■

4   A Set of Distortion Inequalities

Theorem 6 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ and let*

$$[(1-\lambda)\sigma(m,2)\beta_2 + \lambda\sigma(n,2)\gamma_2] \le [(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k].$$

Then, we have for $|z| = r < 1$,

$$r - \frac{B-A}{2\delta_2}r^2 \le |f(z)| \le r + \frac{B-A}{2\delta_2}r^2 \tag{4.1}$$

and

$$1 - \frac{B-A}{\delta_2}r \le |f'(z)| \le 1 + \frac{B-A}{\delta_2}r. \tag{4.2}$$

The results are sharp for the function

$$f(z) = z - \frac{B-A}{2\delta_2}r^2, \tag{4.3}$$

where $\delta_2 = 2(1+B)\left[(1-\lambda)\sigma(m,2)\beta_2 + \lambda\sigma(n,2)\gamma_2\right].$

Proof.   Since $k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]$ is an increasing function of k $(k \ge 2)$, and $f(z) \in Q_m^n(\Phi, \Psi, \lambda, A, B)$, by Theorem 1, we have

$$2\left[(1-\lambda)\sigma(m,2)\beta_2 + \lambda\sigma(n,2)\gamma_2\right]\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty} k\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]a_k$$

$$\le \frac{B-A}{1+B}.$$

That is

$$\sum_{k=2}^{\infty} a_k \le \frac{B-A}{2(1+B)\left[(1-\lambda)\sigma(m,2)\beta_2 + \lambda\sigma(n,2)\gamma_2\right]} = \frac{B-A}{2\delta_2}.$$

$$|f(z)| \le z + \sum_{k=2}^{\infty} |a_k||z|^k$$

$$\le r + \sum_{k=2}^{\infty} a_k r^k \le r + r^2 \sum_{k=2}^{\infty} a_k$$

$$\le r + r^2\left[\frac{B-A}{2(1+B)\left[(1-\lambda)\sigma(m,2)\beta_2 + \lambda\sigma(n,2)\gamma_2\right]}\right]$$

$$= r + r^2\left(\frac{B-A}{2\delta_2}\right) \text{ and}$$

$$|f(z)| \ge r - \sum_{k=2}^{\infty} a_k r^k \ge r - r^2 \sum_{k=2}^{\infty} a_k = r - r^2\left(\frac{B-A}{2\delta_2}\right)$$

Hence (4.1) follows.

Also, in view of the inequality (2.1), we have

$$[(1 - \lambda)\sigma(m, 2)\beta_2 + \lambda\sigma(n, 2)\gamma_2] \sum_{k=2}^{\infty} ka_k \leq \frac{B - A}{1 + B},$$

which gives, $\sum_{k=2}^{\infty} ka_k \leq \dfrac{B - A}{\delta_2}$. Thus

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} ka_k r^{k-1} \leq 1 + r\sum_{k=2}^{\infty} ka_k \leq 1 + \frac{B - A}{\delta_2}r.$$

Similarly we can prove the other inequality $|f'(z)| \geq 1 - \frac{B-A}{\delta_2}r$.

Hence (4.2) follows also. ■

5   Integral Operator

Theorem 7 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$, and let d be a real number such that $d > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{d + 1}{z^d} \int_0^z t^{d-1} f(t)dt \tag{5.1}$$

also belongs to the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$.

Proof.   From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} \left(\frac{d + 1}{d + k}\right) a_k z^k, \tag{5.2}$$

Therefore,

$$\sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] \left(\frac{d + 1}{d + k}\right) a_k$$

$$\leq \sum_{k=2}^{\infty} k\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right] a_k \leq \frac{B - A}{1 + B}.$$

Hence, $F \in Q_m^n(\Phi, \Psi, \lambda, A, B)$. ■

Theorem 8 *Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k \ (a_k \geq 0)$ be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$ and let d be a real number such that $d > -1$. Then the function $f(z)$ defined by (5.1) is univalent in $|z| < R^*$, where*

$$R^* = \inf_k \left[\frac{(1 + B)\left[(1 - \lambda)\sigma(m, k)\beta_k + \lambda\sigma(n, k)\gamma_k\right](d + 1)}{(B - A)(d + m)}\right]^{1/k-1} \quad (k \geq 2). \tag{5.3}$$

The result is sharp.

Proof.   From (5.1), we have

$$f(z) = \frac{z^{1-d}(z^d F(z))'}{d+1} \quad (d > -1)$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{d+k}{d+1} \right) a_k z^k.$$

In order to obtain the required result it suffices to show that

$$|f'(z) - 1| < 1, \tag{5.4}$$

for $|z| < R^*$. Now

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(d+k)}{(d+1)} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$, if $\displaystyle\sum_{k=2}^{\infty} \frac{k(d+k)}{(d+1)} a_k |z|^{k-1} < 1$.

Hence by Theorem 1, (5.4) is true if

$$\frac{k(d+k)}{(d+1)}|z|^{k-1} \leq \frac{k(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]}{B - A}$$

or if

$$|z| < \left[\frac{(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right](d+1)}{(B-A)(d+k)} \right]^{1/k-1}, \quad k \geq 2$$

which proves that f is univalent in $|z| < R^*$.

Sharpness follows if we take,

$$f(z) = z - \frac{(B-A)(d+k)}{k(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right](d+1)} z^k \quad k \geq 2.$$

∎

6   Radii of Close-to-convexity, Starlikeness and Convexity

Theorem 9 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1 \equiv r_1(m, n, \Phi, \Psi, \lambda, A, \rho)$, where*

$$r_1 = \inf_k \left[\frac{(1-\rho)(1+B)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]}{(B-A)} \right]^{1/k-1} \quad k \geq 2. \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ is given by (2.4).

Proof. We must show that $|f'(z) - 1| \leq 1 - \rho$ for $|z| < r_1$. We have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Hence, $|f'(z) - 1| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1 - \rho} \right) a_k |z|^{k-1} \leq 1. \tag{6.2}$$

By Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{k(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_k}{(B-A)} \leq 1.$$

Hence, (6.2) will be true if

$$\frac{k|z|^{k-1}}{1-\rho} \leq \frac{k(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_k}{(B-A)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(B+1)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right] a_k}{(B-A)} \right]^{1/k-1}, \quad k \geq 2. \tag{6.3}$$

The theorem follows easily from (6.3). ∎

Theorem 10 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$, then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2 \equiv r_2(m, n, \Phi, \Psi, \lambda, A, B, \rho)$ where*

$$r_2 = \inf_k \left[\frac{(1-\rho)k(1+B)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]}{(B-A)(k-\rho)} \right]^{1/k-1} \quad (k \geq 2). \tag{6.4}$$

The result is sharp with the extremal function $f(z)$ given by (2.4).

Proof. It is sufficient to show $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$ for $|z| < r_2$. By making use of the inequality,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty}a_k|z|^{k-1}} \tag{6.5}$$

we get, $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{1 - \rho} \leq 1. \tag{6.6}$$

Hence, by using (6.3), (6.6) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \leq \frac{k(1+B)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]}{(B-A)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)k(1+B)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right]}{(k-\rho)(B-A)}\right]^{1/k-1}, \quad k \geq 2. \quad (6.7)$$

The theorem follows easily from (6.7). ∎

Corollary 3 *Let the function $f(z)$ defined by (1.7) be in the class $Q_m^n(\Phi, \Psi, \lambda, A, B)$, then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3 \equiv r_3(m, n, \Phi, \Psi, \lambda, A, B, \rho)$, where*

$$r_3 = \inf_k \left[\frac{(1-\rho)\left[(1-\lambda)\sigma(m,k)\beta_k + \lambda\sigma(n,k)\gamma_k\right](1+B)}{k(k-\rho)(B-A)}\right]^{1/k-1}, \quad (k \geq 2).$$

The result is sharp with the extremal function $f(z)$ given by (2.4).

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New results in the stability study of non-autonomous evolution equations in Banach spaces[1]

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ABSTRACT: The work extends some previous results of the first author, results concerning the Lyapunov stability of the zero solution of some nonautonomous nonlinear evolution equation.

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1 Introduction

In previous works of the first author [5], [6], a nonlinear evolution problem in a Banach space $(X, \| \|)$, of the form

$$
\begin{aligned}
\dot{x} &= Ax + R(t, x), \\
x(0) &= a,
\end{aligned}
\tag{1}
$$

is considered, and the existence of the solution in a neighborhood of $x = 0$ as well as the Lyapunov stability of the null solution are studied via the implicit operator theorem.

In [5] the operator $A : D(A) \subset X \to X$ ($D(A)$ dense in X) is a closed linear operator that generates a strongly continuous, exponentially decreasing semigroup $T(t)$ on X, while the nonlinear operator R is continuous, $R(t, 0) = 0$ for all $t \in \mathbb{R}^+$ and for some $\beta > 0$, $C > 0$ the inequality
$\|R(t, x_1) - R(t, x_2)\| \le C \max^{\beta}(\|x_1\|, \|x_2\|)\|x_1 - x_2\|$ holds for all $t \in \mathbb{R}^+$, and x_1, x_2 in a centered in 0 ball of X.

The hypothesis on the nonlinear operator R is weakened in [6], where the constant C in the above inequality is replaced with a continuous function $C(.)$ with exponential growth. In both papers above it is proved that, for an initial condition with small enough norm, the problem has an unique mild solution (solution of some integral equations associated to the differential abstract equation) and the null solution is exponentially stable in the class of mild solutions. If, moreover, a Hölder condition in t is imposed to R, then the above conclusions hold for the classical solutions also.

2   Problem 1.

We consider the nonautonomous problem

$$\dot{x} = Ax + R(t,x), \quad t \in \mathbb{R}, \tag{2}$$

with the initial condition

$$x(s) = a, \tag{3}$$

for some fixed $s \in \mathbb{R}$.

In [7] the corresponding autonomous problem $(R(t,x) = R(x))$ was considered.

2.1   Hypotheses

I. $A : D(A) \subset X \mapsto X$, *is a closed linear operator, $D(A)$ is dense in X.*

II. *The null space $\mathcal{V} = N(A)$ of the operator A is nontrivial, closed, and X may be written as the direct sum of $\mathcal{V}$ and $\mathcal{U} = R(A)$ $(X = N(A) \oplus R(A))$.*

Hence for any $x \in X$, there is an unique decomposition $x = u + v$ with $u \in \mathcal{U}$, $v \in \mathcal{V}$. The mapping P from X onto $\mathcal{U}$, given by $P(x) = u$, is continuous.

The restriction of the operator A to the subspace $\mathcal{U}$ is the generator of the exponentially decreasing semigroup $U(t)$ of class C_0 (there exist the constants $M > 0$ and $\alpha > 0$ such that for all $t \in \mathbb{R}^+ = [0, +\infty)$ the inequality $||U(t)|| \le M exp(-\alpha t)$ is fulfilled).

III. *The nonlinear mapping R, defined on the Cartesian product of $\mathbb{R}$ with a neighborhood of 0 in X, is continuous, $R(t,0) = 0$ for all $t \in \mathbb{R}$ and there is a $\beta > 0$ and a continuous function $C(t) > 0$ such that for all $t \in \mathbb{R}$ and x_1, x_2 in a neighborhood of 0 in X, the inequality*

$$||R(t,x_1) - R(t,x_2)|| \le C(t) \max\left(||x_1||^{\beta}, ||x_2||^{\beta}\right)||x_1 - x_2|| \tag{4}$$

holds, where $C(\cdot) \in L^1(\mathbb{R})$ hence there is a $\widetilde{C} > 0$ such that, for every $s \in \mathbb{R}$

$$\int_s^{\infty} |C(\theta)| d\theta \le \widetilde{C}.$$

We remark that the kernel of A may be infinite dimensional, hence A is not necessarily a Fredholm operator. We do not insist here on the conditions on A such

that the above hypotheses hold. A comprehensive study of linear semigroups is given in [1].

By hypothesis II, $I - P$ maps X on $N(A)$ and it is continuous. We denote, for simplicity, $\|P\|_{\mathcal{L}(X)}$ by $\|P\|$.

We set in (2) $x(t) = u(t) + v(t)$, where $u(t) = Px(t)$, $v(t) = (I - P)x(t)$ and we project the problem (2), (3) on U and V. We get the system of Cauchy problems

$$\dot{u} = PAu + PR(t, u + v), \quad u(s) = Pa, \tag{5}$$

$$\dot{v} = (I - P)Au + (I - P)R(t, u + v), \quad v(s) = (I - P)a. \tag{6}$$

We consider the system of integral equations

$$u(t) = U(t - s)Pa + \int_s^t U(t - \theta)PR(\theta, u(\theta) + v(\theta))d\theta, \tag{7}$$

$$v(t) = (I - P)a + \int_s^t (I - P)[Au(\theta) + R(\theta, u(\theta) + v(\theta))]d\theta. \tag{8}$$

If the pair $(u(t), v(t))$ is a solution of the integral equations (7)-(8) then $x(t) = u(t) + v(t)$ is called the mild solution of the Cauchy problem (2), (3).

Hypothesis II implies $(I - P)A = 0$ and the equations above become

$$u(t) = U(t - s)Pa + \int_s^t U(t - \theta)PR(\theta, u(\theta) + v(\theta))d\theta, \tag{9}$$

$$v(t) = (I - P)a + \int_s^t (I - P)R(\theta, u(\theta) + v(\theta))d\theta. \tag{10}$$

We set $t = \tau + s$ and we perform the change of variables $\theta' = \theta - s$, to obtain

$$u(\tau + s) = U(\tau)Pa + \int_0^\tau U(\tau - \theta')PR(\theta' + s, u(\theta' + s) + v(\theta' + s))d\theta', \tag{11}$$

$$v(\tau + s) = (I - P)a + \int_0^\tau (I - P)R(\theta' + s, u(\theta' + s) + v(\theta' + s))d\theta'. \tag{12}$$

For every $s \in \mathbb{R}$, and any function $x : [s, \infty) \mapsto X$ we define the function $x|_s : [0, \infty) \mapsto X$, by $x|_s(\theta) = x(s + \theta)$.

For a space $X_1 \subseteq X$, we consider the space $\mathcal{C}_b([0, \infty), X_1)$ of continuous bounded functions defined on $[0, \infty)$ with values in X_1, with the supremum norm ($\|x\|_0 = \sup_{t \geq 0} \|x(t)\|$), and denote it shortly by $\mathcal{C}_b(X_1)$.

We define the operators

$$D_1 : X \mapsto C_b(\mathcal{U}), \ a \mapsto D_1(a), \ D_1(a)(\tau) = U(\tau)Pa, \ \tau \geq 0;$$

$$D_2 : X \mapsto C_b(\mathcal{V}), \ a \mapsto D_2(a), \ D_2(a)(\tau) = (I - P)a;$$

$$F_{1s} : \mathcal{C}_b(\mathcal{X}) \mapsto \mathcal{C}_b(\mathcal{U}), \ F_{1s}(x)(\tau) = \int_0^\tau U(\tau - \theta) PR(s + \theta, x(\theta)) d\theta,$$

where $x = u + v$, and

$$F_{2s} : \mathcal{C}_b(\mathcal{X}) \mapsto \mathcal{C}_b(\mathcal{V}), \ F_{2s}(x)(\tau) = \int_0^\tau (I - P) R(s + \theta, x(\theta)) d\theta.$$

For these operators the following relations hold:

$$\|D_1\|_{\mathcal{L}(X, \mathcal{C}_b(\mathcal{U}))} \leq M \|P\|,$$

$$\|D_2\|_{\mathcal{L}(X, \mathcal{C}_b(\mathcal{V}))} \leq 1 + \|P\|,$$

$$\|F_{1s}(x)(\tau)\| \leq \int_0^\tau M e^{-\alpha(\tau - \theta)} \|P\| C(s + \theta) \|x(\theta)\|^{\beta+1} d\theta \leq M \widetilde{C} \|P\| \|x\|_0^{\beta+1},$$

$$\|F_{2s}(x)(\tau)\| \leq \widetilde{C}(1 + \|P\|) \|x\|_0^{\beta+1}.$$

The two integral equations (11), (12) may be written as the equation

$$x|_s = D(a) + F_s(x|_s), \tag{13}$$

where

$$D = D_1 + D_2, \ F_s = F_{1s} + F_{2s}.$$

Equation (13) may be regarded as a fixed point problem in $\mathcal{C}_b(X)$,

$$\phi = \Phi(a, \phi) \tag{14}$$

where

$$\Phi(a, \phi) = D(a) + F_s(\phi). \tag{15}$$

We remark that $\phi^*(a)$ is a fixed point of $\Phi(a, .)$ if and only if the function $x(\cdot; s, a) : [s, \infty) \mapsto X$, defined by $x(s + \theta; s, a) = \phi^*(\theta, a)$, $\theta \geq 0$, is a mild solution of problem (2), (3).

2.2   Results

Theorem 1.   *There are two positive numbers r_0, r_1, such that for $\|a\| \leq r_0$, the mapping $\Phi(a, \cdot)$ given by (15) is an uniform (with respect to a) contraction from $\overline{B}(0, r_1) \subset \mathcal{C}_b(X)$ to itself.*

Proof. We have, for any $\phi_1, \phi_2 \in \mathcal{C}_b(X)$

$$\|\Phi(a, \phi_1) - \Phi(a, \phi_2)\|_0 = \|F_s(\phi_1) - F_s(\phi_2)\|_0 \leq$$

$$\leq \|F_{1s}(\phi_1) - F_{1s}(\phi_2)\|_0 + \|F_{2s}(\phi_1) - F_{2s}(\phi_2)\|_0.$$

We have

$$\|F_{1s}(\phi_1) - F_{1s}(\phi_2)\|_0 \leq \int_s^t M e^{-\alpha(t - \theta)} \|PR(s + \theta, \phi_1(\theta)) - PR(s + \theta, \phi_2(\theta))\| d\theta \leq$$

$$\leq \int_s^t Me^{-\alpha(t-\theta)}\|P\|C(s+\theta)\max(\|\phi_1(\theta)\|^\beta, \|\phi_2(\theta)\|^\beta)\|\phi_1(\theta)-\phi_2(\theta)\|d\theta \leq$$

$$\leq M\widetilde{C}\|P\|\max\{\|\phi_1\|_0^\beta, \|\phi_2\|_0^\beta\}\|\phi_1-\phi_2\|_0, \ (\forall)\phi_1, \phi_2 \in \mathcal{C}_b(X), \tag{16}$$

and

$$\|F_{2s}(\phi_1)-F_{2s}(\phi_2)\|_0 \leq \int_0^\tau \|(I-P)R(s+\theta, \phi_1(\theta)) - (I-P)R(s+\theta, \phi_2(\theta))\|d\theta$$

$$\leq \int_0^\tau (1+\|P\|)|C(s+\theta)|d\theta \max\{\|\phi_1\|_0^\beta, \|\phi_2\|_0^\beta\}\|\phi_1-\phi_2\|_0$$

$$\leq \widetilde{C}(1+\|P\|)\max\{\|\phi_1\|_0^\beta, \|\phi_2\|_0^\beta\}\|\phi_1-\phi_2\|_0.$$

For $\phi_1, \phi_2 \in B(0,r) \subset \mathcal{C}_b(X)$, we have

$$\|F_s(\phi_1)-F_s(\phi_2)\|_0 \leq \widetilde{C}[(M+1)\|P\|+1]r^\beta\|\phi_1-\phi_2\|_0.$$

Let $r_1 > 0$ be such that $r_1 < \left\{\widetilde{C}[(M+1)\|P\|+1]\right\}^{-1/\beta}$. On the sphere $\overline{B}(0,r_1) \subset \mathcal{C}_b(X)$, the mapping $\Phi(a, \cdot)$ is an uniform (with respect to a) contraction.

If $\|a\| \leq r_0$, $\|\phi\|_0 \leq r_1$, then

$$\|\Phi(a,\phi)\|_0 \leq [(M+1)\|P\|+1]r_0 + \widetilde{C}[(M+1)\|P\|+1]r_1^{\beta+1}.$$

By imposing to this last quantity to be less than r_1, we find the restriction

$$\|a\| \leq r_0 := \frac{r_1}{(M+1)\|P\|+1}\left\{1 - \widetilde{C}[(M+1)\|P\|+1]r_1^\beta\right\}.$$

Hence, for $\|a\| \leq r_0$, Φ is an uniform contraction on $\overline{B}(0,r_1)$. $\square$

The uniform Banach contraction principle implies that, for $\|a\| \leq r_0$, the fixed point problem (14) has an unique solution $\varphi^*(a) \in \mathcal{C}_b(X)$.

Hence the function $x(\cdot; s, a)$ given by $x(s+\theta; s, a) = \varphi^*(\theta, a)$, $\theta \geq 0$ is a mild solution of problem (2), (3). Since Φ is an uniform contraction with respect to a, the fixed point φ^* is continuous with respect to a. Hence the function that maps $a \in X$ to $x(s+\cdot; s, a) \in \mathcal{C}_b(X)$ is continuous. From here and from $x(.; s, 0) = 0$, the Lyapunov stability of the null solution in the class of mild solutions follows, where the Lyapunov stability is understood in the following sense

Definition 1. [2] *A classical (resp. -mild) solution $x(.; s, a)$ of problem (2), (3) is called stable if for every $\epsilon > 0$ and every $s' > s$ there is a $\delta = \delta(\epsilon, s')$ such that for every $y \in X$ with $\|y - x(s'; s, a)\| \leq \delta$, the classical (resp. -mild) solution $x(\cdot; s', y)$ exists, is defined on $[s', \infty)$, and*

$$\|x(t; s, a) - x(t; s', y)\| < \epsilon$$

for every $t \geq s'$.

If a Hölder condition with respect to time is added on R, then we obtain the existence of classical solutions and also the stability of the 0 solution in the class of classical solutions.

Remark. If we assume also that $R(t, v) = 0$ for every $v \in \mathcal{V}$, then by observing that in this case $\|R(\theta, u(\theta) + v(\theta))\| \leq C(\theta)\|u(\theta)\|^{\beta+1}$ holds, and by the reasoning of [5] we obtain that $\|u(t)\|$ tends exponentially to 0 when $t \to \infty$.

3   Problem 2.

We consider a perturbation of Problem 1, that is

$$\dot{x} = Ax + Bx + R(t, x), \quad t \in \mathbb{R}, \tag{17}$$

with the initial condition

$$x(s) = a. \tag{18}$$

Here B is a linear bounded operator $B : X \mapsto R(A)$ with norm such that

$$L := \frac{M}{\alpha} \|P\| \|B\|_{\mathcal{L}(X)} < 1. \tag{19}$$

Due to the range of B, only the first equation of the projected equations differs from those of the previous section. The projected equations are, in this case,

$$\dot{u} = PAu + PB(u + v) + PR(t, u + v), \quad u(s) = Pa, \tag{20}$$

$$\dot{v} = (I - P)R(t, u + v), \quad v(s) = (I - P)a. \tag{21}$$

We consider the integral equations

$$u(t) = U(t-s)Pa + \int_s^t U(t-\theta)PBu(\theta)d\theta + \int_s^t U(t-\theta)PR(\theta, u(\theta) + v(\theta))d\theta, \tag{22}$$

$$v(t) = (I - P)a + \int_s^t (I - P)R(\theta, u(\theta) + v(\theta))d\theta. \tag{23}$$

We say that a function $x(\cdot) = u(\cdot) + v(\cdot)$ is an A-mild solution of equation (17) if $(u(\cdot), v(\cdot))$ is the solution of (22)-(23). We named this solution A-mild because the semigroup generated only by A (and not by $A + B$) is used in the first integral equation. We proceed as for Problem 1, only that we define also the operator

$$E : \mathcal{C}_b(\mathcal{X}) \mapsto \mathcal{C}_b(\mathcal{U}), \; E(x)(t) = \int_s^t U(t - \theta)PB(x(\theta))d\theta,$$

for which

$$\|E(\phi)\|_0 \le L\|\phi\|_0,$$

with L given by (19) and

$$\|E(\phi_1) - E(\phi_2)\|_0 \le L\|\phi_1 - \phi_2\|_0.$$

The integral equations above are equivalent with the fixed point problem in $\mathcal{C}_b(X)$

$$\phi = \Phi(a, \phi),$$

where

$$\Phi(a, \phi) = D(a) + E(\phi) + F_s(\phi). \tag{24}$$

Theorem 2. *There are two positive numbers r_0, r_1, such that for $\|a\| \leq r_0$, the mapping $\Phi(a, \cdot)$ given by (24) is an uniform (with respect to a) contraction from $\overline{B}(0, r_1) \subset \mathcal{C}_b(X)$ to itself.*

Proof. The computations in the proof of Theorem 1 and the properties of E lead to

$$\|\Phi(a, \phi_1) - \Phi(a, \phi_2)\|_0 \leq \left\{ \widetilde{C}[(M+1)\|P\| + 1]r^\beta + L \right\} \|\phi_1 - \phi_2\|_0$$

for $\|\phi_i\|_0 \leq r$, $i = 1, 2$.

We choose a $r_1 > 0$ such that $r_1 < \left(\frac{1-L}{\widetilde{C}[(M+1)\|P\|+1]} \right)^{1/\beta}$.

On the closed sphere $\|\phi\|_0 \leq r_1$ the mapping Φ is a contraction (uniform with respect to a). We have

$$\|\Phi(a, \phi_1)\|_0 \leq [(M+1)\|P\| + 1]\|a\| + \widetilde{C}[(M+1)\|P\| + 1]r_1^{\beta+1} + Lr_1.$$

We impose the condition

$$[(M+1)\|P\| + 1]\|a\| + \widetilde{C}[(M+1)\|P\| + 1]r_1^{\beta+1} + Lr_1 \leq r_1$$

and we find that this condition is satisfied for

$$\|a\| \leq r_0 := \frac{1}{(M+1)\|P\| + 1} \left\{ 1 - L - \widetilde{C}[(M+1)\|P\| + 1]r_1^\beta \right\} r_1.$$

The assertion of the theorem follows. $\square$

From this point, with the same reasonings as in the preceding section, we obtain that, for $\|a\| \leq r_0$, r_0 defined in the proof of Theorem 2, the problem (17), (18) has an unique A-mild solution $x(\cdot; s, a)$ in the space of continuous bounded functions defined from $[s, \infty)$ to X. Moreover, from the continuity of the function that maps $a \in X$ to $x(s + \cdot; s, a) \in \mathcal{C}_b(X)$ the Lyapunov stability of the null solution in the class of A-mild solutions follows.

If the nonlinear map $R(.,.)$ satisfies a Hölder condition with respect to time, then the above assertions are valid in the classical sense also.

Remark. If we assume also that $R(t, v) = 0$ for $v \in \mathcal{V}$, then by observing that $\|R(\theta, u(\theta) + v(\theta))\| \leq C(\theta)\|u(\theta)\|^{\beta+1}$, and by reasonings similar to those of Section 4 of [3] we obtain that $\|u(t)\| \to 0$ when $t \to \infty$.

4 Problem 3.

We consider again equation (2) with condition (3), but here we assume that only Hypothesis I on A and Hypothesis III on R are fulfilled (with no special hypotheses on the kernel of A). Moreover, we assume that A generates a bounded semigroup $\{T(t)\}_{t \geq 0}$ of operators on X, that is for a $M > 0$, $\|T(t)\|_{\mathcal{L}(X)} \leq M$, $\forall t \geq 0$.

We consider the integral equation

$$x(t) = T(t - s)a + \int_s^t T(t - \theta)R(\theta, x(\theta))d\theta. \tag{25}$$

A solution of this equation is called a mild solution of problem (2), (3). Obviously, any classical solution is a mild solution of this problem. As before, we successively transform the integral equation (25) into a fixed point problem:

$$x(s + \tau) = T(\tau)a + \int_0^\tau T(\tau - \theta')R(s + \theta', x(s + \theta'))d\theta',$$

hence

$$x|_s(\tau) = T(\tau)a + \int_0^\tau T(\tau - \theta')R(s + \theta', x|_s(\theta'))d\theta'. \tag{26}$$

Now, we define $D : X \mapsto \mathcal{C}_b(X), \ a \to D(a)(.), D(a)(\tau) = T(\tau)a$, and $F_s : \mathcal{C}_b(X) \mapsto \mathcal{C}_b(X)$

$$F_s(\phi) = \int_0^\tau T(\tau - \theta')R(s + \theta', \phi(\theta'))d\theta',$$

for which we have $\|D(a)\|_0 \le M\|a\|$, and, respectively

$$\|F_s(\phi)\|_0 \le \int_0^\tau MC(s + \theta')\|\phi(\theta')\|^{\beta+1}d\theta' \le M\widetilde{C}\|\phi\|_0^{\beta+1},$$

hence F_s takes indeed values in $\mathcal{C}_b(X)$. Moreover,

$$\|F_s(\phi_1) - F_s(\phi_2)\|_0 \le \int_0^\tau MC(s + \theta') \max(\|\phi_1(\theta')\|^\beta, \|\phi_2(\theta')\|^\beta)\|\phi_1(\theta') - \phi_2(\theta')\|d\theta' \le$$

$$\le M\widetilde{C} \max(\|\phi_1\|_0^\beta, \|\phi_2\|_0^\beta)\|\phi_1 - \phi_2\|_0.$$

We define the mapping $\Phi : X \times \mathcal{C}_b(X) \mapsto \mathcal{C}_b(X)$, given by

$$\Phi(a, \phi) = D(a) + F_s(\phi) \tag{27}$$

and remark that equation (26) is equivalent to the fixed point problem in $\mathcal{C}_b(X)$

$$\phi = \Phi(a, \phi). \tag{28}$$

Now consider a positive number r_1 such that $M\widetilde{C}r_1^\beta < 1$. On the ball $B(0, r_1)$ the mapping Φ is a contraction (uniform with respect to a). Now we take an a such that

$$M\|a\| + M\widetilde{C}r_1^{\beta+1} \le r_1 \Leftrightarrow \|a\| \le r_0 := r_1(1 - M\widetilde{C}r_1^\beta)/M.$$

We thus proved

Theorem 3. *There are two positive numbers r_0, r_1, such that for $\|a\| \le r_0$, the mapping $\Phi(a, \cdot)$ given by (27) is an uniform (with respect to a) contraction from $\overline{B}(0, r_1) \subset \mathcal{C}_b(X)$ to itself.*

Hence, if $\|a\| < r_0$, there is an unique solution of equation (28) in $\overline{B}(0, r_1)$. Thus an unique mild solution $x(.; s, a)$ of problem (2), (3) exists. Then, the local Lyapunov stability of the mild 0 solution follows from the continuity of the function $a \in X \to x(s + \cdot; s, a) \in \mathcal{C}_b(X)$.

If we add a Hölder condition in t on R the local stability of 0 as a classical solution follows also.

4.1   Problem 4.

We show in this section that similar results may be obtained also in the case when the linear operator is a function of time, that is for the equation

$$\dot{x} = A(t)x + R(t, x), \ t \in \mathbb{R}, \tag{29}$$

with the initial condition

$$x(s) = a.$$

Here $A(t) : D \subset X \mapsto X$ are closed linear operators, defined on the domain D (independent of t), dense in X, and we assume that the problem

$$\dot{u} = A(t)u, \ u(s) = a \tag{30}$$

has an unique classical solution, that is a function $u \in \mathcal{C}([s, \infty), X) \cap \mathcal{C}^1([s, \infty), X)$ such that $u(t) \in D$ for $t > s$ and (30) is satisfied. In this situation by the relation

$$U(t, s)u(s) = u(t)$$

an *evolution family* is generated.

An evolution family is a family of linear bounded operators $\{U(t, s), t \geq s, s \in \mathbb{R}\}$ that satisfy

$$U(t, r)U(r, s) = U(t, s), \ U(s, s) = I.$$

We say that an evolution family is *strongly continuous* if the mapping $(t, s) \to U(t, s)$ is strongly continuous on the set $t, s \in \mathbb{R}, t > s$.

The most difficult problem concerning this case is that of giving sufficient conditions in order that the linear problem attached to the above problem generates such an evolution family. This is not the subject of this paper and we indicate the work [4] and the references therein.

The evolution family is named *exponentially bounded* if the inequality

$$\|U(t, s)\|_{\mathcal{L}(X)} \leq M e^{\omega(t-s)},$$

$M \geq 1$, $\omega \in \mathbb{R}$, holds. If $\omega < 0$, the evolution family is named *exponentially stable.*

We first assume that the evolution family generated by (30) is exponentially stable hence there is a positive α such that

$$\|U(t, s)\|_{\mathcal{L}(X)} \leq M e^{-\alpha(t-s)}. \tag{31}$$

In order to study the existence of the solution of problem (29), we consider the integral equation

$$x(t) = U(t, s)a + \int_s^t U(t, \theta)R(\theta, x(\theta))d\theta. \tag{32}$$

Any classical solution of problem (29) is a solution of (32). A solution of (32) is called a *mild solution* of (29). From this point, formally, the reasoning may be lead as in [6], since only the norm estimates and the hypotheses on R will be used.

In [6] the hypotheses on R differ from Hypothesis III from the beginning of this work only in what concerns the function $C(\cdot)$. More precisely, it is assumed that one of the inequalities

$$C(t)e^{-\gamma\beta} \leq C^*, \ \text{ for a } \gamma \in (0, \alpha), \ C^* > 0;$$

$$\int_0^\infty C(s)e^{-\alpha\beta s}ds < \infty$$

holds (hence $C(.)$ may be unbounded).

As in [6], the problem may be brought to the form of a fixed point problem in the space $\mathcal{C}_\gamma(X) = \{x : [0, \infty) \mapsto X \mid \sup_{[0,\infty)} \|x(t)\|e^{\gamma t} < \infty\}$. It follows that for sufficient small $\|a\|$ there is an unique mild solution of (29) and that the null solution is locally asymptotically stable (in the class of mild solutions), in the sense given by

Definition 2. [2] *A classical (resp. -mild) solution $x(.; s, a)$ of problem (2), (3) is called asymptotically stable if it is stable and, for every $s' > s$, there is a $\delta = \delta(s') > 0$ such that for $y \in X$ with $\|y - x(s'; s, a)\| < \delta$, the classical (resp. -mild) solution $x(\cdot; s', y)$ exists, is defined on $[s', \infty)$, and*

$$\|x(t; s, a) - x(t; s', y)\| \to 0$$

as $t \to \infty$.

If we assume that the evolution family is only bounded ($\alpha = 0$), then with the same reasonings as in Section 3 of the present paper, we obtain that for sufficient small $\|a\|$ there is an unique mild solution of (29) and that the null solution is Lyapunov stable (in the class of mild solutions).

If the function $R(\cdot, \cdot)$ satisfies also a Hölder condition in t the above conclusions are valid in the classical sense also.

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[6] D. Beck, *Introduction to Dynamical System*, Progr. Math. 54, Birkhäuser, Basel, 1978.

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