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Perturbation Theory, M-essential Spectra of 2×2 Operator Matrices and Application to Transport Operators

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Abstract: In this article we give some results on perturbation theory of 2×2 block operator matrices on the product of Banach spaces. Furthermore, we investigate their M-essential spectra. Finally, we apply the obtained results to determine the M-essential spectra of two group transport operators with general boundary conditions in the Banach space $L_p([-a,a] \times [-1,1]) \times L_p([-a,a] \times [-1,1])$, $p \geq 1$ and $a > 0$.

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1. Introduction

Let X and Y be two Banach spaces. In this work we will discuss some results on perturbation theory of 2×2 operator matrices on $X \times Y$ and we will investigate their M-essential spectra. We consider operators in the following form

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are, in general, unbounded operators. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts between these spaces. Note that, in general L_0 is neither a closed nor a closable operator, even if its entries are closed. In [1], F. V. Atkinson,

H. Langer, R. Mennicken and A. A. Shkalikov give some sufficient conditions under which L_0 is closable and describe its closure, that we will denote by L.

In recent years, number of papers have been devoted to study the essential spectra of block operator matrices acting in a product of Banach spaces, (see [1], [2], [4], [5], [6], [14], [16], [23] and [26]). Most authors, there, have proposed methods for dealing with spectral theory for operators in the form $L_0 - \mu M$ where $M = I$. We note that the idea of studying the spectral characteristics of the 2×2 matrix operator goes back to the classics of the spectral theory for the differential operator. Hence several analysis focused on this issue may be found in the literature, see for example [10], [12], [13], [17], [18], [19], [20] and [25]. Recently, C. Tretter gives in [21], [22] and [23] an account research and presents a wide panorama of methods to investigate the spectral theory of block operator matrices. In the paper [8], M. Faierman, R. Mennicken and M. Möller propose a method for dealing with the spectral theory for pencils of the form $L_0 - \mu M$, where M is a bounded operator.

In this work, we generalize the results of [16] where M-essential spectra of some 2×2 operator matrices on $X \times X$ are discussed with $M = I$. For this, first we establish some results on perturbation theory of 2×2 operator matrices, essentially we prove the following result:

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2) \text{ if and only if } F_{ij} \in \mathcal{F}^b(X_j, X_i),\ \ i,j = 1,2,$$

where $\mathcal{F}^b(X_j, X_i)$ designs the set of Fredholm perturbations (see Definition 2.2). Then we pursue the analysis started in [8] and we determine the M-essential spectra of a 2×2 matrix operator where M is a bounded operator defined on the product of two Banach spaces $X \times Y$.

We organize the paper in the following way: In Section 2, some preliminary abstract results about Fredholm operators are given. In Section 3, we establish some results on perturbation theory of 2×2 operator matrices. The Section 4 is devoted to the study of the M-essential spectra of a 2×2 matrix operator. Finally, in Section 5 we apply the obtained results to investigate the M-essential spectra of a two-group transport operator on L_p-spaces, $1 \leq p < \infty$.

2. Preliminary results

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp. $\mathcal{C}(X,Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y. For $T \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in Y.

Let S be a bounded operator from X to Y. For $T \in \mathcal{C}(X,Y)$, we define the

S-resolvent set of T by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ has a bounded inverse}\},$$

and the S-spectrum of T by:

$$\sigma_S(T) = \mathbb{C} \setminus \rho_S(T).$$

Now, we introduce the following important operator classes:

The set of upper semi-Fredholm operators is defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by:

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denote the set of Fredholm operators from X into Y and $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ the set of semi-Fredholm operators from X into Y. While the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T, for $T \in \Phi(X,Y)$. We say that the complex number λ is in $\Phi_{+T,S}$, $\Phi_{-T,S}$, $\Phi_{\pm T,S}$ or $\Phi_{T,S}$ if $\lambda S - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $C(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi_\pm(X)$, respectively.

In this paper we are concerned with the following S-essential spectra:

$$\begin{aligned}
&\sigma_{e_1,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_+(X,Y)\} := \mathbb{C} \setminus \Phi_{+T,S},\\
&\sigma_{e_2,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_-(X,Y)\} := \mathbb{C} \setminus \Phi_{-T,S},\\
&\sigma_{e_3,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi_\pm(X,Y)\} := \mathbb{C} \setminus \Phi_{\pm T,S},\\
&\sigma_{e_4,S}(T) := \{\lambda \in \mathbb{C} \quad \text{such that} \quad \lambda S - T \notin \Phi(X,Y)\} := \mathbb{C} \setminus \Phi_{T,S},\\
&\sigma_{e_5,S}(T) := \mathbb{C} \setminus \rho_{5,S}(T),\\
&\sigma_{e_6,S}(T) := \mathbb{C} \setminus \rho_{6,S}(T),
\end{aligned}$$

where $\rho_{5,S}(T) := \{\lambda \in \Phi_{T,S} \text{ such that } i(\lambda S - T) = 0\}$ and $\rho_{6,S}(T)$ denote the set of those $\lambda \in \rho_{5,S}(T)$ such that all scalars near λ are in $\rho_S(T)$. They can be ordered as

$$\sigma_{e_3,S}(T) = \sigma_{e_1,S}(T) \cap \sigma_{e_2,S}(T) \subset \sigma_{e_4,S}(T) \subset \sigma_{e_5,S}(T) \subset \sigma_{e_6,S}(T).$$

Note that if $S = I$, we recover the usual definition of the essential spectra of a closed densely defined linear operator (see [16]).

Let us, now, introduce some notation and then continue with some lemmas and propositions.

Proposition 2.1. *[15] Let $T \in \mathcal{C}(X,Y)$ and consider S a nonzero bounded linear operator from X into Y. Then we have the following results:*
(i) $\Phi_{T,S}$ is open.
(ii) $i(\lambda S - T)$ is constant on any component of $\Phi_{T,S}$.
(iii) $\alpha(\lambda S - T)$ and $\beta(\lambda S - T)$ are constant on any component of $\Phi_{T,S}$ except on a discrete set of points on which they have larger values.

Definition 2.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. F is called strictly singular, if for every infinite-dimensional closed subspace $\mathcal{M}$ of X, the restriction of F to $\mathcal{M}$ is not bijective.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X to Y.

Definition 2.2. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.
(i) The operator F is called Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$.
(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp. $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp. $U \in \Phi_-(X,Y)$).

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbations and by $\mathcal{F}_+(X,Y)$ (resp. $\mathcal{F}_-(X,Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations.

Remark 2.1. Let $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$ respectively. If in Definition 2.2 we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$.

The sets of Fredholm perturbations and semi-Fredholm perturbations were introduced in [9]. In particular it is shown that $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X) := \mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X) := \mathcal{F}^b_-(X,Y)$ are closed two-sided ideals of $\mathcal{L}(X)$.

In general, we have the following inclusions:

$$\mathcal{K}(X,Y) \subset \mathcal{S}(X,Y) \subset \mathcal{F}^b(X,Y).$$

Note that, the set $\mathcal{F}^b(X,Y)$ can strictly contains $\mathcal{S}(X,Y)$. Indeed, in [27], the author gives some geometric conditions on the Banach spaces for which the equality $\mathcal{S}(X,Y) = \mathcal{F}^b(X,Y)$ does not hold.
Recall the following result established in [3].

Lemma 2.1. *[3] Let X and Y be two Banach spaces, then*

$$\mathcal{F}(X,Y) = \mathcal{F}^b(X,Y).$$

Proposition 2.2. *[15] Let T_1, T_2 are two closed densely defined linear operators on X and S an invertible operator on X.*

(i) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b(X)$, then

$$\sigma_{e_i,S}(T_1) = \sigma_{e_i,S}(T_2), \ i = 4, 5.$$

(ii) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b_+(X)$, then

$$\sigma_{e_1,S}(T_1) = \sigma_{e_1,S}(T_2).$$

(iii) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}^b_-(X)$, then

$$\sigma_{e_2,S}(T_1) = \sigma_{e_2,S}(T_2).$$

(iv) If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$*, the operator* $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{F}_+^b(X) \cap \mathcal{F}_-^b(X)$*, then*

$$\sigma_{e_3,S}(T_1) = \sigma_{e_3,S}(T_2).$$

Definition 2.3. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to have a left Fredholm inverse if there exists an operator $T_l \in \mathcal{L}(Y,X)$ such that $T_lT - I \in \mathcal{K}(X)$. Similarly, T is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y,X)$ such that $TT_r - I \in \mathcal{K}(Y)$. The operators T_l and T_r are called, respectively, left and right Fredholm inverse of T.

We will denote by $\Phi_l^b(X,Y)$ (resp. $\Phi_r^b(X,Y)$) the set of bounded operators which have left Fredholm inverse (resp. right Fredholm inverse).
It follows from [17, Theorems 14. and 15. p. 160] that

$$\Phi_l^b(X,Y) = \{T \in \Phi_+^b(X,Y) \text{ such that } R(T) \text{ is complemented}\}$$

and

$$\Phi_r^b(X,Y) = \{T \in \Phi_-^b(X,Y) \text{ such that } \ker(T) \text{ is complemented}\},$$

where a subspace $N \subset X$ is said to be complemented if there exists a closed subspace $M \subset X$ such that $N \oplus M = X$.
Note that we have the following inclusions:

$$\Phi^b(X,Y) \subset \Phi_l^b(X,Y) \subset \Phi_+^b(X,Y)$$

and

$$\Phi^b(X,Y) \subset \Phi_r^b(X,Y) \subset \Phi_-^b(X,Y).$$

Definition 2.4. Let X and Y be two Banach spaces. We denote by

$$\mathcal{F}_l^b(X,Y) = \{F \in \mathcal{L}(X,Y) \text{ such that } T + F \in \Phi_l^b(X,Y) \text{ whenever } T \in \Phi_l^b(X,Y)\}$$

and

$$\mathcal{F}_r^b(X,Y) = \{F \in \mathcal{L}(X,Y) \text{ such that } T + F \in \Phi_r^b(X,Y) \text{ whenever } T \in \Phi_r^b(X,Y)\}.$$

The set $\mathcal{F}_l^b(X,X)$ (resp. $\mathcal{F}_r^b(X,X)$) will be denoted by $\mathcal{F}_l^b(X)$ (resp. $\mathcal{F}_r^b(X)$).

Proposition 2.3. *Let* X*,* Y *and* Z *be three Banach spaces.*
(i) If $A \in \Phi^b(Y,Z)$ *and* $T \in \Phi_l^b(X,Y)$ *(resp.* $T \in \Phi_r^b(X,Y)$*), then* $AT \in \Phi_l^b(X,Z)$ *(resp.* $AT \in \Phi_r^b(X,Z)$*).*

(ii) If $A \in \Phi^b(X,Y)$ *and* $T \in \Phi_l^b(Y,Z)$ *(resp.* $T \in \Phi_r^b(Y,Z)$*), then* $TA \in \Phi_l^b(X,Z)$ *(resp.* $TA \in \Phi_r^b(X,Z)$*).*

Proof. (i) Let $A \in \Phi^b(Y,Z)$, then, by [24, Theorem 5.4.] there exist $A_0 \in \mathcal{L}(Z,Y)$ and $K_1 \in \mathcal{K}(Y)$ (resp. $K_2 \in \mathcal{K}(Z)$) such that $A_0A = I_Y - K_1$ (resp. $AA_0 = I_Z - K_2$). On the other hand, there exist $T_l \in \mathcal{L}(Y,X)$ (resp. $T_r \in \mathcal{L}(Y,X)$) and $K_3 \in \mathcal{K}(X)$ (resp. $K_4 \in \mathcal{K}(Y)$) such that $T_lT = I_X - K_3$ (resp. $TT_r = I_Y - K_4$) since $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$). So, $T_lA_0AT = I_X - K_3 - T_lK_1T$ (resp. $ATT_rA_0 = I_Z - K_4 - AK_2A_0$), which imply that $AT \in \Phi_l^b(X,Z)$ (resp. $AT \in \Phi_r^b(X,Z)$).
(ii) The proof is analogous to the previous one. □

Proposition 2.4. *Let X, Y and Z be three Banach spaces.*
(i) *If the set $\Phi^b(Y,Z)$ is not empty, then*

$$F_1 \in \mathcal{F}_l^b(X,Y) \textit{ and } A \in \Phi^b(Y,Z), \textit{ imply } AF_1 \in \mathcal{F}_l^b(X,Z),$$

$$F_1 \in \mathcal{F}_r^b(X,Y) \textit{ and } A \in \Phi^b(Y,Z), \textit{ imply } AF_1 \in \mathcal{F}_r^b(X,Z).$$

(ii) *If the set $\Phi^b(X,Y)$ is not empty, then*

$$F_2 \in \mathcal{F}_l^b(Y,Z) \textit{ and } A \in \Phi^b(X,Y), \textit{ imply } F_2A \in \mathcal{F}_l^b(X,Z),$$

$$F_2 \in \mathcal{F}_r^b(Y,Z) \textit{ and } A \in \Phi^b(X,Y), \textit{ imply } F_2A \in \mathcal{F}_r^b(X,Z).$$

Proof. (i) Since $A \in \Phi^b(Y,Z)$, then there exist $A_0 \in \mathcal{L}(Z,Y)$ and $K \in \mathcal{K}(Z)$ such that $AA_0 = I_Z - K$. By [24, Theorem 5.5. p. 105] we have $A_0 \in \Phi^b(Z,Y)$. Let $B \in \Phi_l^b(X,Z)$ (resp. $B \in \Phi_r^b(X,Z)$). Using the Propriety 2.3(i) we deduce that $A_0B \in \Phi_l^b(X,Y)$ (resp. $A_0B \in \Phi_r^b(X,Y)$). Then $A_0B + F_1 \in \Phi_l^b(X,Y)$ (resp. $A_0B + F_1 \in \Phi_r^b(X,Y)$). And so $AF_1 + B - KB \in \Phi_l^b(X,Y)$ (resp. $AF_1 + B - KB \in \Phi_r^b(X,Y)$). Therefore $AF_1 + B \in \Phi_l^b(X,Y)$ (resp. $AF_1 + B \in \Phi_r^b(X,Y)$).
(ii) The proof of (ii) is obtained as like as the proof of (i). □

Theorem 2.1. *Let X, Y and Z be Banach spaces.*
(i) *If the set $\Phi^b(Y,Z)$ is not empty, then*

$$F_1 \in \mathcal{F}_l^b(X,Y) \textit{ and } A \in \mathcal{L}(Y,Z), \textit{ imply } AF_1 \in \mathcal{F}_l^b(X,Z),$$

$$F_1 \in \mathcal{F}_r^b(X,Y) \textit{ and } A \in \mathcal{L}(Y,Z), \textit{ imply } AF_1 \in \mathcal{F}_r^b(X,Z).$$

(ii) *If the set $\Phi^b(X,Y)$ is not empty, then*

$$F_2 \in \mathcal{F}_l^b(Y,Z) \textit{ and } A \in \mathcal{L}(X,Y), \textit{ imply } F_2A \in \mathcal{F}_l^b(X,Z),$$

$$F_2 \in \mathcal{F}_r^b(Y,Z) \textit{ and } A \in \mathcal{L}(X,Y), \textit{ imply } F_2A \in \mathcal{F}_r^b(X,Z).$$

Remark 2.2. It follows from Definition 2.4 and the previous theorem that $\mathcal{F}_l^b(X)$ and $\mathcal{F}_r^b(X)$ are two-sided ideals of $\mathcal{L}(X)$.

Proof of Theorem 2.1. (i) Let $C \in \Phi^b(Y, Z)$ and $\lambda \in \mathbb{C}$. We denote by $A_1 = A - \lambda C$ and $A_2 = \lambda C$. For sufficiently large λ, using [24, Theorem 5.11], we have $A_1 \in \Phi^b(Y, Z)$. It follows from Proposition 2.4(i) that $A_1F_1 \in \mathcal{F}_l^b(X, Z)$ (resp. $A_1F_1 \in \mathcal{F}_r^b(X, Z)$) and $A_2F_1 \in \mathcal{F}_l^b(X, Z)$ (resp. $A_2F_1 \in \mathcal{F}_r^b(X, Z)$). This implies $A_1F_1 + A_2F_1 = AF_1 \in \mathcal{F}_l^b(X, Z)$ (resp. $A_1F_1 + A_2F_1 = AF_1 \in \mathcal{F}_r^b(X, Z)$). (ii) We can check the other results in the same way us the previous one. □

Proposition 2.5. *Let X and Y be two Banach spaces. If the set $\Phi^b(Y, Z)$ is not empty, then we have the inclusions:*

$$\mathcal{K}(X, Y) \subset \mathcal{F}_l^b(X, Y) \subset \mathcal{F}^b(X, Y),$$

$$\mathcal{K}(X, Y) \subset \mathcal{F}_r^b(X, Y) \subset \mathcal{F}^b(X, Y).$$

Proof. We will prove the first result. The same reasoning remains valid for the second one. It is obvious that $\mathcal{K}(X, Y) \subset \mathcal{F}_l^b(X, Y)$. For the second inclusion, let $F \in \mathcal{F}_l^b(X, Y)$ and consider $A \in \Phi^b(X, Y)$, then there exist $A_0 \in \mathcal{L}(Y, X)$ and $K \in \mathcal{K}(X)$ such that $A_0A = I_X - K$. So, $A_0(A + F) = I_X - K + A_0F$. It follows from Theorem 2.1 that $A_0F \in \mathcal{F}_l^b(X)$, then $A_0(A + F) \in \Phi_l^b(X)$. Using the inclusion $\Phi_l^b(X, Y) \subset \Phi_+^b(X, Y)$, we obtain $A + F \in \Phi_+^b(X, Y)$. On the other hand, consider the mapping φ defined by: $\forall \lambda \in \mathbb{C}, \varphi(\lambda) = A + \lambda F$. Note that φ is continuous and $\varphi([0, 1]) \subset \Phi_+^b(X, Y)$, using Proposition 2.1, we can deduce that $i(A+F) = i(A) < \infty$. Hence $A + F \in \Phi^b(X, Y)$. □

3. Some results on perturbation theory of 2×2 matrix operator

In this section we will establish some results on perturbation theory of 2×2 matrix operator that acts on a product of Banach spaces X_1 and X_2. The following lemmas are necessary.

Lemma 3.1. *Let $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and consider the 2×2 matrix operator $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where $C \in \mathcal{L}(X_2, X_1)$. Then*

(i) If $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $M_C \in \Phi^b(X_1 \times X_2)$, for every $C \in \mathcal{L}(X_2, X_1)$.

(ii) If $A \in \Phi_+^b(X_1)$ and $B \in \Phi_+^b(X_2)$, then $M_C \in \Phi_+^b(X_1 \times X_2)$ for every $C \in \mathcal{L}(X_2, X_1)$.

(iii) If $A \in \Phi_-^b(X_1)$ and $B \in \Phi_-^b(X_2)$, then $M_C \in \Phi_-^b(X_1 \times X_2)$ for every $C \in \mathcal{L}(X_2, X_1)$.

Proof. (i) Write M_C in the form

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{3.1}$$

Since $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ are both Fredholm operators. So, M_C is a Fredholm operator, since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible for every $C \in \mathcal{L}(X_2, X_1)$.

(ii) and (iii) can be checked in the same way as (i). □

Remark 3.1. Using the same reasoning as the proof of the previous lemma we can show that:

(i) If $A \in \Phi^b(X_1)$ and $B \in \Phi^b(X_2)$, then $M_D := \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ is a Fredholm operator on $X_1 \times X_2$ for every $D \in \mathcal{L}(X_1, X_2)$.

(ii) If $A \in \Phi^b_+(X_1)$ and $B \in \Phi^b_+(X_2)$, then $M_D \in \Phi^b_+(X_1 \times X_2)$ for every $D \in \mathcal{L}(X_1, X_2)$.

(iii) If $A \in \Phi^b_-(X_1)$ and $B \in \Phi^b_-(X_2)$, then $M_D \in \Phi^b_-(X_1 \times X_2)$ for every $D \in \mathcal{L}(X_1, X_2)$.

Lemma 3.2. *Let $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and consider the 2×2 matrix operator $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where $C \in \mathcal{L}(X_2, X_1)$.*

(i) If $M_C \in \Phi^b_+(X_1 \times X_2)$, then $A \in \Phi^b_+(X_1)$.

(ii) If $M_C \in \Phi^b_-(X_1 \times X_2)$, then $B \in \Phi^b_-(X_2)$.

Proof. The result follows immediately from Eq. (3.1). □

Remark 3.2. (i) It follows immediately from the last Lemma that if $M_C \in \Phi^b(X_1 \times X_2)$, then $A \in \Phi^b_+(X_1)$ and $B \in \Phi^b_-(X_2)$.

(ii) Using the same reasoning as the proof of the previous lemma we can show that if the operator $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ is in $\Phi^b(X_1 \times X_2)$ for some $D \in \mathcal{L}(X_1, X_2)$, then $A \in \Phi^b_-(X_1)$ and $B \in \Phi^b_+(X_2)$.

Theorem 3.1. *Let $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ where $F_{ij} \in \mathcal{L}(X_j, X_i)$, $i, j = 1, 2$. Then*

$$F \in \mathcal{F}^b(X_1 \times X_2) \text{ if and only if } F_{ij} \in \mathcal{F}^b(X_j, X_i),\ \forall i, j = 1, 2.$$

Remark 3.3. (i) It follows from Lemma 2.1 that Theorem 3.1 remains valid if we replace $\mathcal{F}^b(X_1 \times X_2)$ by $\mathcal{F}(X_1 \times X_2)$ and $\mathcal{F}^b(X_j, X_i)$ by $\mathcal{F}(X_j, X_i)$, $i, j = 1, 2$.

(ii) It is sufficient to apply the definition of compact and strictly singular operators to verify that the result of Theorem 3.1 is true for these classes of operators. Therefore, in view of Remark 2.1 the previous theorem may be viewed as a generalization to a more large class of operators.

Proof. To prove the second implication, we consider the following decomposition,

$$F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_{22} \end{pmatrix}.$$

It is sufficient to prove that if $F_{ij} \in \mathcal{F}^b(X_j, X_i)$, $i, j = 1, 2$ then, each operator in the right hand side of the previous equality is a Fredholm perturbation on $X_1 \times X_2$. We will prove the result for example for the first operator. The proof for the other operators will be in the same way. Consider $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi^b(X_1 \times X_2)$ and denote $\widetilde{F} := \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$. It follows from [17, Theorem 12 p.159] that there exist $L_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$ and $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \in \mathcal{K}(X_1 \times X_2)$ such that

$$LL_0 = I - K \text{ on } X_1 \times X_2.$$

Then,

$$(L + \widetilde{F})L_0 = I - K + \widetilde{F}L_0 = \begin{pmatrix} I - K_{11} + F_{11}A_0 & -K_{12} + F_{11}B_0 \\ -K_{21} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}^b(X_1)$ and using Theorem 2.1(ii), we will have $I - K_{11} + F_{11}A_0 \in \Phi^b(X_1)$. This, with the fact that $I - K_{22} \in \Phi^b(X_2)$, we can deduce from Lemma 3.1(i) that $(L + \widetilde{F})L_0 - \begin{pmatrix} 0 & 0 \\ -K_{21} & 0 \end{pmatrix}$ is a Fredholm operator on $X_1 \times X_2$. The fact that K_{21} is a compact operator and $L_0 \in \Phi^b(X_1 \times X_2)$ leads, by [24, Theorem 5.13], to $L + \widetilde{F} \in \Phi^b(X_1 \times X_2)$.

Conversely, assume that $F \in \mathcal{F}^b(X_1 \times X_2)$. We will prove that $F_{11} \in \mathcal{F}^b(X_1)$. Let $A \in \Phi^b(X_1)$ and define the operator $L_1 := \begin{pmatrix} A & -F_{12} \\ 0 & I \end{pmatrix}$. It follows, from Lemma 3.1(i) that $L_1 \in \Phi^b(X_1 \times X_2)$. Thus $F + L_1 = \begin{pmatrix} A + F_{11} & 0 \\ F_{21} & I + F_{22} \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. The use of Remark 3.2(ii) leads to

$$A + F_{11} \in \Phi^b_-(X_1). \tag{3.2}$$

In the same way, we consider the Fredholm operator $\begin{pmatrix} A & 0 \\ -F_{21} & I \end{pmatrix}$ and we use Remarks 3.1(i) and 3.2(i) to deduce that

$$A + F_{11} \in \Phi^b_+(X_1). \tag{3.3}$$

It follows from Eqs. (3.2) and (3.3) that

$$F_{11} \in \mathcal{F}^b(X_1).$$

In the same way, we prove that $F_{22} \in \mathcal{F}^b(X_2)$.

Now, we will prove that $F_{12} \in \mathcal{F}^b(X_2, X_1)$ and $F_{21} \in \mathcal{F}^b(X_1, X_2)$. For this, consider $A \in \Phi^b(X_2, X_1)$ and $B \in \Phi^b(X_1, X_2)$. Then $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. Using the fact that $F_{11} \in \mathcal{F}^b(X_1)$, $F_{22} \in \mathcal{F}^b(X_2)$ and the result of the second implication, we deduce that $F + \begin{pmatrix} -F_{11} & 0 \\ 0 & -F_{22} \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2)$. Hence, $\begin{pmatrix} 0 & A + F_{12} \\ B + F_{21} & 0 \end{pmatrix} \in \Phi^b(X_1 \times X_2)$. So, $A + F_{12} \in \Phi^b(X_2, X_1)$ and $B + F_{21} \in \Phi^b(X_1, X_2)$. □

Theorem 3.2. *Let* $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ *where* $F_{ij} \in \mathcal{L}(X_j, X_i)$, $i, j = 1, 2$. *Then*
(i) $F \in \mathcal{F}_l^b(X_1 \times X_2)$ *if and only if* $F_{ij} \in \mathcal{F}_l^b(X_j, X_i)$, $\forall i, j = 1, 2$.
(ii) $F \in \mathcal{F}_r^b(X_1 \times X_2)$ *if and only if* $F_{ij} \in \mathcal{F}_r^b(X_j, X_i)$, $\forall i, j = 1, 2$.

Proof. (i) Using the same notations as in the proof of Theorem 3.1 we obtain:

$$L_0(L + \widetilde{F}) = I - K + L_0\widetilde{F} = \begin{pmatrix} I - K_{11} + A_0F_{11} & -K_{12} \\ -K_{21} + C_0F_{11} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}_l^b(X_1)$ and using Theorem 2.1(i), we deduce that $I - K_{11} + A_0F_{11} \in \Phi_l^b(X_1)$. So, there exist an operator $H \in \mathcal{L}(X_1 \times X_2)$ and $K_0 \in \mathcal{K}(X_1)$ such that $H(I - K_{11} + A_0F_{11}) = I - K_0$. Therefore,

$$\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) = I - \begin{pmatrix} K_0 & HK_{12} \\ K_{21} & K_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C_0F_{11} & 0 \end{pmatrix}.$$

Using Theorem 2.1(i), Proposition 2.5(i) and Theorem 3.1 we obtain $\begin{pmatrix} 0 & 0 \\ C_0F_{11} & 0 \end{pmatrix} \in \mathcal{F}^b(X_1 \times X_2)$, and so, $\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) \in \Phi^b(X_1 \times X_2)$, then there exist $L_1 \in \mathcal{L}(X_1 \times X_2)$ and $\widetilde{K} \in \mathcal{K}(X_1 \times X_2)$ such that $L_1 \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} L_0(L + \widetilde{F}) = I - \widetilde{K}$, which implies that $\widetilde{F} \in \mathcal{F}_b^l(X_1 \times X_2)$.
(ii) We prove this assertion in the same way as in (i). □

Remark 3.4. The following questions remain open:
(i) $F \in \mathcal{F}_+^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_+^b(X_j, X_i)$, $\forall i, j = 1, 2$.
(ii) $F \in \mathcal{F}_-^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_-^b(X_j, X_i)$, $\forall i, j = 1, 2$.

4. The M-essential spectra of 2×2 matrix operator

The purpose of this section is to discuss the M-essential spectra of the 2×2 matrix operator L, closure of L_0 that acts on the Banach space $X \times Y$ where M is a bounded operator formally defined on the product space $X \times Y$ by a matrix

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and L_0 is given by

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The operator A acts on X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts on X (resp. on Y).

In what follows, we will assume that the following conditions hold:

(H_1) A is a closed, densely defined linear operator on X with nonempty M_1-resolvent set $\rho_{M_1}(A)$.

(H_2) The operator B is densely defined linear operator on X and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $(A - \mu M_1)^{-1}B$ is closable. (In particular, if B is closable then $(A - \mu M_1)^{-1}B$ is closable).

(H_3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $C(A - \mu M_1)^{-1}$ is bounded.

(H_4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in Y, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $D - C(A - \mu M_1)^{-1}B$ is closable, we will denote by $S(\mu)$ the closure of the operator $D - (C - \mu M_3)(A - \mu M_1)^{-1}(B - \mu M_2)$.

Remark 4.1. (i) It follows from the closed graph theorem that the operator $G(\mu) := (A - \mu M_1)^{-1}(B - \mu M_2)$ is bounded on Y.

(ii) We emphasize that neither the domain of $S(\mu)$ nor the property of being closable depend on μ. Indeed, consider $\lambda, \mu \in \rho_{M_1}(A)$, then we have:

$$S(\lambda) - S(\mu) = (\lambda - \mu)\left[M_3 G(\mu) + F(\lambda)M_2 + F(\lambda)M_1 G(\mu)\right], \tag{4.1}$$

where $F(\lambda) = (C - \lambda M_3)(A - \lambda M_1)^{-1}$. Since the operators $F(\lambda)$ and $G(\mu)$ are bounded, then the difference $S(\lambda) - S(\mu)$ is bounded. Therefore neither the domain of $S(\mu)$ nor the property of being closable depend on μ.

We recall the following result established in [8] which describes the closure of the operator L_0.

Theorem 4.1. *[8] Let conditions* $(H_1) - (H_3)$ *be satisfied and the lineal* $\mathcal{D}(B) \cap \mathcal{D}(D)$ *be dense in* X*. Then the operator* L_0 *is closable if and only if the operator* $D - C(A - \mu M_1)^{-1}B$ *is closable in* X*, for some* $\mu \in \varrho_{M_1}(A)$*. Moreover, the closure* L *of* L_0 *is given by*

$$L = \mu M + \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} A - \mu M_1 & 0 \\ 0 & S(\mu) - \mu M_4 \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{4.2}$$

Lemma 4.1. *(i) If $M_3 \in \mathcal{F}^b(X,Y)$ and $F(\lambda) \in \mathcal{F}^b(X,Y)$, for some $\lambda \in \rho_{M_1}(A)$, then $F(\lambda) \in \mathcal{F}^b(X,Y)$ for all $\lambda \in \rho_{M_1}(A)$.*

(ii) If $M_2 \in \mathcal{F}^b(Y,X)$ and if $G(\lambda) \in \mathcal{F}^b(Y,X)$, for some $\lambda \in \rho_{M_1}(A)$, then $G(\lambda) \in \mathcal{F}^b(Y,X)$ for all $\lambda \in \rho_{M_1}(A)$.

(iii) If $F(\lambda)$, $G(\lambda)$, M_2 and M_3 are Fredholm perturbations, for some $\lambda \in \rho_{M_1}(A)$, then $\sigma_{e_i,M_4}(S(\lambda))$ does not depend on $\lambda \in \rho_{M_1}(A)$, for $i = 1, ..., 6$.

Proof. (i) The result follows from the identity

$$F(\lambda) - F(\mu) = (\lambda - \mu)\big[F(\lambda)M_1 - M_3\big](A - \mu M_1)^{-1}, \text{ for all } \lambda \text{ and } \mu \in \rho_{M_1}(A).$$

(ii) The result follows from the identity

$$G(\lambda) - G(\mu) = (\lambda - \mu)(A - \lambda M_1)^{-1}\big[M_1 G(\mu) - M_2\big], \text{ for all } \lambda \text{ and } \mu \in \rho_{M_1}(A).$$

(iii) The result of this assertion follows from Eq. (4.1). □

In the sequel, we will denote the complement of a subset $\Omega \subset$ C by ${}^C\Omega$.

Theorem 4.2. *Let L_0 be the 2×2 matrix operator satisfying conditions $(H_1)-(H_4)$. If M_2 and M_3 are Fredholm perturbations and if for some (hence for all) $\mu \in \rho_{M_1}(A)$, $F(\mu)$ and $G(\mu)$ are Fredholm perturbations, then*

$$\sigma_{e_4,M}(L) = \sigma_{e_4,M_1}(A) \cup \sigma_{e_4,M_4}(S(\mu)).$$

and

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)).$$

Moreover, if ${}^C\sigma_{e_4,M_1}(A)$ is connected, then

$$\sigma_{e_5,M}(L) = \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)).$$

If in addition, ${}^C\sigma_{e_5,M}(L)$ is connected, $\rho_M(L) \neq \emptyset$, ${}^C\sigma_{e_5,M_4}(S(\mu))$ is connected and $\rho_{M_4}(S(\mu)) \neq \emptyset$, then

$$\sigma_{e_6,M}(L) = \sigma_{e_6,M_1}(A) \cup \sigma_{e_6,M_4}(S(\mu)).$$

Proof. Let $\mu \in \rho_{M_1}(A)$ be such that the operators $F(\mu)$ and $G(\mu)$ are Fredholm perturbations and set $\lambda \in \mathbb{C}$. While writing $\lambda M - L = \mu M - L + (\lambda - \mu)M$, using the relation (4.2) we have

$$\lambda M - L = UV(\lambda)W - (\lambda - \mu)\begin{pmatrix} 0 & M_1 G(\mu) - M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1 G(\mu) \end{pmatrix} \tag{4.3}$$

where $U = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}$, $W = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$ and $V(\lambda) = \begin{pmatrix} \lambda M_1 - A & 0 \\ 0 & \lambda M_4 - S(\mu) \end{pmatrix}$. Since the operators $F(\mu)$, $G(\mu)$, M_2 and M_3 are Fredholm perturbations, then by Theorem 3.1 the second operator in the right hand side of Eq.(4.3) is a Fredholm perturbation. So $\lambda M - L$ is a Fredholm operator if and only if $UV(\lambda)W$ is a Fredholm operator. Now, observe that the operators U and W are bounded and have bounded inverse, hence the operator $UV(\lambda)W$ is a Fredholm operator if and only if $V(\lambda)$ has this property if and only if $\lambda M_1 - A$ (resp. $\lambda M_4 - S(\mu)$) is a Fredholm operator on X (resp. on Y) and

$$i(\lambda M - L) = i(\lambda M_1 - A) + i(\lambda M_4 - S(\mu)). \tag{4.4}$$

Therefore,

$$\sigma_{e_4,M}(L) = \sigma_{e_4,M_1}(A) \cup \sigma_{e_4,M_4}(S(\mu))$$

and

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)). \tag{4.5}$$

Suppose now that ${}^{C}\sigma_{e_4,M_1}(A)$ is connected. By assumption (H_1), $\rho_{M_1}(A)$ is not empty. Let $\alpha \in \rho_{M_1}(A)$, then, $\alpha M_1 - A \in \Phi(X)$ and $i(\alpha M_1 - A) = 0$. Since $\rho_{M_1}(A) \subseteq \rho_{4,M_1}(A)$ and by Proposition 2.1, $i(\lambda M_1 - A)$ is constant on any component of $\Phi_{M_1,A}$, then $i(\lambda M_1 - A) = 0$ for all $\lambda \in \rho_{4,M_1}(A)$. It follows, immediately, from Eqs (4.4) and (4.5) that

$$\sigma_{e_5,M}(L) = \sigma_{e_5,M_1}(A) \cup \sigma_{e_5,M_4}(S(\mu)). \tag{4.6}$$

Assume further, that ${}^{C}\sigma_{e_5,M_1}(A)$ is connected. Then, by Lemma 2.1 in [15] and using Eq. 4.6 we have

$$\sigma_{e_6,M}(L) = \sigma_{e_6,M_1}(A) \cup \sigma_{e_6,M_4}(S(\mu)). \qquad \square$$

In the sequel we will denote, for $\mu \in \varrho_{M_1}(A)$, by $M(\mu)$ the following operator

$$M(\mu) = \begin{pmatrix} 0 & M_1 G(\mu) - M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1 G(\mu) \end{pmatrix}.$$

Theorem 4.3. *(i) If the operator $M(\mu) \in \mathcal{F}_+(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then*

$$\sigma_{e_1,M}(L) = \sigma_{e_1,M_1}(A) \cup \sigma_{e_1,M_4}(S(\mu)).$$

(ii) If the operator $M(\mu) \in \mathcal{F}_-(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then

$$\sigma_{e_2,M}(L) = \sigma_{e_2,M_1}(A) \cup \sigma_{e_2,M_4}(S(\mu)).$$

(iii) If $M(\mu) \in \mathcal{F}_+(X \times Y) \cap \mathcal{F}_-(X \times Y)$ for some $\mu \in \rho_{M_1}(A)$, then

$$\begin{aligned} \sigma_{e_3,M}(L) &= \sigma_{e_3,M_1}(A) \cup \sigma_{e_3,M_4}(S(\mu)) \cup [\sigma_{e_2,M_1}(A) \cup \sigma_{e_1,M_4}(S(\mu))] \\ &\quad \cup [\sigma_{e_1,M_1}(A) \cup \sigma_{e_2,M_4}(S(\mu))]. \end{aligned}$$

Proof. The assertions (i) and (ii) follow immediately from Eq. (4.3).
The assertion (iii) is an immediate consequence of (i) and (ii). $\square$

Remark 4.2. Theorems (4.2) and (4.3) generalize the Theorem (3.2) in [16].

5. Application to two-group transport operators

The aim of this section is to apply the obtained results to study the M-essential spectra of a class of linear two-group transport operators on L_p-spaces, $1 \leq p < \infty$, with abstract boundary conditions.

Let

$$X_p := L_p((-a,a) \times (-1,1); dxdv), \quad a > 0,\ 1 \leq p < \infty.$$

We consider the following two-group transport operators with abstract boundary conditions:

$$A_H = T_H + K$$

where

$$T_H \psi = \begin{pmatrix} T_{H_1} & 0 \\ 0 & T_{H_2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with $K_{ij}, i,j = 1,2$, are bounded linear operators defined on X_p by

$$\begin{cases} K_{ij}: & X_p & \longrightarrow & X_p \\ & u & \longrightarrow & K_{ij}u(x,v) = \displaystyle\int_{-1}^{1} \kappa_{ij}(x,v,v')u(x,v')dv' \end{cases} \tag{5.1}$$

and the kernels $\kappa_{ij} : (-a,a) \times (-1,1) \times (-1,1) \longrightarrow \mathrm{R}$ are assumed to be measurable. Each operator T_{H_j}, $j = 1,2$, is defined by

$$\begin{cases} T_{H_j} : \mathcal{D}(T_{H_j}) \subset X_p \longrightarrow X_p, \\ \qquad\qquad \varphi \qquad\qquad \longrightarrow (T_{H_j}\varphi)(x,v) = -v\dfrac{\partial \varphi}{\partial x}(x,v) - \sigma_j(v)\varphi(x,v), \\ \mathcal{D}(T_{H_j}) = \{\varphi \in W \text{ such that } \varphi^i = H_j \varphi^o\}, \end{cases}$$

where W is the space defined by

$$W = \{\varphi \in X_p \text{ such that } v\frac{\partial \varphi}{\partial x} \in X_p\}$$

and $\sigma_j(.) \in L^\infty(-1,1)$. φ^o, φ^i represent the outgoing and the incoming fluxes related by the boundary operator H_j ("o" for the outgoing and "i" for the incoming) and given by

$$\begin{cases} \varphi^i(v) = \varphi(-a,v), & v \in (0,1), \\ \varphi^i(v) = \varphi(a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(-a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(a,v), & v \in (0,1). \end{cases}$$

We denote by X_p^o and X_p^i the following boundary spaces:

$X_p^o := L_p\left[\{-a\} \times (-1,0); |v|dv\right] \times L_p\left[\{a\} \times (0,1); |v|dv\right] := X_{1,p}^o \times X_{2,p}^o$
equipped with the norm

$$\begin{aligned} \|u^o, X_p^o\| &:= \left(\|u_1^o, X_{1,p}^o\|^p + \|u_2^o, X_{2,p}^o\|^p\right)^{\frac{1}{p}} \\ &= \left[\int_{-1}^{0} |u(-a,v)|^p |v| dv + \int_0^1 |u(a,v)|^p |v| dv\right]^{\frac{1}{p}}, \end{aligned}$$

and

$X_p^i := L_p\left[\{-a\} \times (0,1); |v|dv\right] \times L_p\left[\{a\} \times (-1,0); |v|dv\right] := X_{1,p}^i \times X_{2,p}^i$
equipped with the norm

$$\begin{aligned} \|u^i, X_p^i\| &:= \left(\|u_1^i, X_{1,p}^i\|^p + \|u_2^i, X_{2,p}^i\|^p\right)^{\frac{1}{p}} \\ &= \left[\int_0^1 |u(-a,v)|^p |v| dv + \int_{-1}^{0} |u(a,v)|^p |v| dv\right]^{\frac{1}{p}}. \end{aligned}$$

It is well known that any function u in W possesses traces on the spacial boundary $\{-a\} \times (-1,0)$ and $\{a\} \times (0,1)$ which respectively belong to the spaces X_p^o and X_p^i (see, for instance, [7] or [11]). they are denoted, respectively, by u^o and u^i.

It is clear that the operator A_H is defined on $\mathcal{D}(T_{H_1}) \times \mathcal{D}(T_{H_2})$. We will denote the operator A_H by

$$A_H := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{cases} A_{11} = T_{H_1} + K_{11}, \\ A_{12} = K_{12}, \\ A_{21} = K_{21}, \\ A_{22} = T_{H_2} + K_{22}. \end{cases}$$

The object of this part is to determine the M-essential spectra of the operator A_H where M is the following operator

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

with $M_i,\ i = 1,4$ are defined by

$$\begin{cases} M_i: & X_p \longrightarrow X_p \\ & \varphi \longrightarrow (M_i\varphi)(x,v) = \eta_i(v)\varphi(x,v) \end{cases}$$

where $\eta_i(.) \in \mathcal{L}^\infty(-1,1)$ and M_2, M_3 are in $\mathcal{F}(X_p)$.

To verify the hypotheses of Theorem 4.2, we shall first determine the expression of the M_1-resolvent of the operator T_{H_1}. Let $\varphi \in X_p,\ \lambda \in \mathbb{C}$ and consider the M_1-resolvent equation for T_{H_1}

$$(\lambda M_1 - T_{H_1})\psi_1 = \varphi, \tag{5.2}$$

where the unknown ψ_1 must be in $\mathcal{D}(T_{H_1})$. Let

$$\lambda_j^* = \text{ess-inf}\ \sigma_j(v), \quad j = 1,2;$$

$$\mu_j^* = \text{ ess-inf } \eta_j(v), \quad j = 1, 2;$$

we suppose that $\mu_j^* > 0, \quad j = 1, 2$ and let

$$\lambda_0^j := \begin{cases} -\lambda_j^*, & \text{if } \|H_j\| \leq 1 \\ -\dfrac{\lambda_j^*}{\mu_j^*} + \dfrac{1}{2a\mu_j^*} \log(\|H_j\|), & \text{if } \|H_j\| > 1. \end{cases}$$

Therefore, for $\lambda \in \mathbb{C}$ such that $\mu_1^* Re\lambda + \lambda_1^* > 0$, the solution of Eq. (5.2) is formally given by

$$\psi_1(x, v) = \begin{cases} \psi_1(-a, v) e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a+x|}{|v|}} \\ +\dfrac{1}{|v|} \displaystyle\int_{-a}^{x} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}} \varphi(x', v) dx', \quad 0 < v < 1, \\ \psi_1(a, v) e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a-x|}{|v|}} \\ +\dfrac{1}{|v|} \displaystyle\int_{x}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}} \varphi(x', v) dx', \quad -1 < v < 0. \end{cases} \tag{5.3}$$

Accordingly, $\psi_1(a, v)$ and $\psi_1(-a, v)$ are given by

$$\psi_1(a, v) = \psi_1(-a, v) e^{-2a\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}}$$

$$+\frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a-x|}{|v|}} \varphi(x, v) dx, \quad 0 < v < 1, \tag{5.4}$$

$$\psi_1(-a, v) = \psi_1(a, v) e^{-2a\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}} \tag{5.5}$$

$$+\frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a+x|}{|v|}} \varphi(x, v) dx, \quad -1 < v < 0. \tag{5.6}$$

For the clarity of our subsequent analysis, we introduce the following bounded operators:

$$\begin{cases} M_\lambda : X_p^i \to X_p^o, \quad M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \quad \text{with} \\ M_\lambda^+ u(-a, v) := u(-a, v) e^{-2a\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}}, \quad 0 < v < 1, \\ M_\lambda^- u(a, v) := u(a, v) e^{-2a\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}}, \quad -1 < v < 0, \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \to X_p, \quad B_\lambda u := \chi(-1, 0)(v) B_\lambda^- u + \chi(0, 1)(v) B_\lambda^+ u) \quad \text{with} \\ B_\lambda^+ u(x, v) := u(-a, v) e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a+x|}{|v|}}, \quad 0 < v < 1, \\ B_\lambda^- u(x, v) := u(a, v) e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a-x|}{|v|}}, \quad -1 < v < 0, \end{cases}$$

$$\begin{cases} G_\lambda : X_p \to X_p^o, \quad G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \quad \text{with} \\ G_\lambda^+ \varphi(-a, v) := \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a-x|}{|v|}} \varphi(x, v) dx, \quad 0 < v < 1, \\ G_\lambda^- \varphi(a, v) := \frac{1}{|v|} \int_{-a}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|a+x|}{|v|}} \varphi(x, v) dx, \quad -1 < v < 0, \end{cases}$$

and finally, we consider

$$\begin{cases} C_\lambda : X_p \to X_p, \quad C_\lambda\varphi := \chi(-1,0)C_\lambda^-\varphi + \chi(0,1)C_\lambda^+\varphi \quad \text{with} \\ C_\lambda^+\varphi(x,v) := \frac{1}{|v|}\int_{-a}^{x} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)dx', \quad 0 < v < 1, \\ C_\lambda^-\varphi(x,v) := \frac{1}{|v|}\int_{x}^{a} e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)dx', \quad -1 < v < 0, \end{cases}$$

where $\chi_{(0,1)}(.)$ and $\chi_{(-1,0)}(.)$ denote the characteristic functions of the intervals $(-1,0)$ and $(0,1)$, , respectively. The operators M_λ, B_λ, G_λ and C_λ are bounded by : $e^{-2a\mu^* Re\lambda}$, $(p\mu^* Re\lambda)^{-1/p}$, $(\mu^* Re\lambda)^{-1/q}$, respectively, where q denotes the conjugate of p and $(\mu^* Re\lambda)^{-1}$.

Lemma 5.1. (i) *If $\kappa_{ij}(x,v,v')$ defines a regular operator, then $(\lambda M_1 - T_{H_1})^{-1}K_{ij}$ is compact on X_p, for $1 < p < \infty$ and weakly compact on X_1, $i,j = 1,2$.*
(ii) *If $\kappa_{ij}(x,v,v')/|v'|$ defines a regular operator, then $K_{ij}(\lambda M_1 - T_{H_1})^{-1}$ is weakly compact on X_1, $i,j = 1,2$.*

Proof. (i) This assertion was proved in [15].
(ii) The proof of this assertion is a straightforward adaption from Lemma 4.2 in [16]. □

Theorem 5.1. *If $\kappa_{21}(x,v,v')$ (resp. $\kappa_{21}(x,v,v')/|v'|$) defines a regular operator, then the operators $F(\lambda) := (K_{21} - \lambda M_3)(A_{11} - \lambda M_1)^{-1}$ and $G(\lambda) := (A_{11} - \lambda M_1)^{-1}(K_{12} - \lambda M_2)$ are Fredholm perturbations on X_p, $1 \le p < \infty$.*

Proof. It follows from Remark 3.1.(ii) in [15] that there exists $\lambda \in \rho_{M_1}(T_{H_1})$ such that
$r_\sigma\left((T_{H_1} - \lambda M_1)^{-1}K_{11}\right) < 1$. For such λ, the equation

$$(K_{11} + T_{H_1} - \lambda M_1)\varphi = \psi$$

may be transformed into

$$\left((T_{H_1} - \lambda M_1)^{-1}K_{11} - I\right)\varphi = (T_{H_1} - \lambda M_1)^{-1}\psi.$$

Then, by the fact that $r_\sigma\left((T_{H_1} - \lambda M_1)^{-1}K_{11}\right) < 1$, we obtain

$$(A_{11} - \lambda M_1)^{-1} = \sum_{n\ge 0}[(T_{H_1} - \lambda M_1)^{-1}K_{11}]^n(T_{H_1} - \lambda M_1)^{-1}.$$

So,

$$F(\lambda) = K_{21}\sum_{n\ge 0}\left[(T_{H_1} - \lambda M_1)^{-1}K_{11}\right]^n(T_{H_1} - \lambda M_1)^{-1} - \lambda M_3(A_{11} - \lambda M_1)^{-1}.$$

Since $M_3 \in \mathcal{F}(X_p)$ and the use of Lemma 5.1 allows us to conclude that $F(\lambda) \in \mathcal{F}(X_p)$.
The same reasoning allows us to prove that $G(\lambda) \in \mathcal{F}(X_p)$. □

Now, we are ready to express the M-essential spectra of two-group transport operators with general boundary conditions.

Theorem 5.2. *If the operators $H_j \in \mathcal{F}(X_p)$, $j = 1, 2,$ $1 \leq p < \infty$ and the operators K_{11}, K_{22}, K_{12} are regular and if in addition $\kappa_{21}(x, v, v')$ (resp. $\kappa_{21}(x, v, v')/|v'|$) defines a regular operator on X_p, for $1 < p < \infty$ (resp. on X_1), then*

$$\sigma_{e_i,M}(A_H) = \{\lambda \in \mathrm{C} \text{ such that } Re\lambda \leq -min(\lambda_1^*, \lambda_2^*)\}, \quad \text{for } i = 1, ..., 6.$$

Proof. Let $\lambda \in \rho_{M_1}(T_{H_1})$ such that $r_\sigma(\lambda M_1 - T_{H_1})K_{11} < 1$, then

$$(\lambda M_1 - A_{11})^{-1} - (\lambda M_1 - T_{H_1})^{-1} = \sum_{n \geq 1} [(\lambda M_1 - T_{H_1})^{-1} K_{11}]^n (\lambda M_1 - T_{H_1})^{-1}.$$

Since K_{11} is regular, then it follows from Lemma 5.1 that the operator $(\lambda M_1 - A_{11})^{-1} - (\lambda M_1 - T_{H_1})^{-1}$ is compact on X_p, for $1 < p < \infty$ and weakly compact on X_1, the use of [15, Theorem 3.3] leads to

$$\sigma_{e_i,M_1}(A_{11}) = \sigma_{e_i,M_1}(T_{H_1}) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_1^*}{\mu_1^*}\right\}, \quad i = 1, ..., 6. \quad (5.7)$$

Let $\mu \in \rho_{M_1}(A_{11})$. The operator $S(\mu)$ is given by

$$S(\mu) = A_{22} - K_{21}G(\mu).$$

By Lemma 5.1, The operator $K_{21}G(\mu)$ is compact on X_p, for $1 < p < \infty$, and weakly compact on X_1, then it follows from Proposition 2.2 that $\sigma_{e_i,M_4}(S(\mu)) = \sigma_{e_i,M_4}(A_{22})$, $i = 1, ..., 6$. By the same reasoning, we have

$$\sigma_{e_i,M_4}(S(\mu)) = \sigma_{e_i,M_1}(A_{22}) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_2^*}{\mu_2^*}\right\}, \quad i = 1, ..., 6. \quad (5.8)$$

Applying Theorem 4.2 and using Eqs (5.7) and (5.8), we get

$$\sigma_{e_i,M}(A_H) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\min\left(\frac{\lambda_1^*}{\mu_1^*}, \frac{\lambda_2^*}{\mu_2^*}\right)\right\}, \quad i = 1, ..., 6.$$ □

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