Journal of
Mathematics
and Applications

JMA No 42, pp 109-116 (2019)

On the Maximum Modulus of a Polynomial

V.K. Jain

Abstract: For a polynomial $p(z)$ of degree n, having no zeros in $|z| < 1$ Ankeny and Rivlin had shown that for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.$$

Using Govil, Rahman and Schmeisser's refinement of the generalization of Schwarz's lemma we have obtained a refinement of Ankeny and Rivlin's result. Our refinement is also a refinement of Dewan and Pukhta's refinement of Ankeny and Rivlin's result.

AMS Subject Classification: 30C10, 30A10.
*Keywords and Phrases:*Maximum modulus; Polynomial; Refinement; Refinement of the generalization of Schwarz's lemma; No zeros in $|z| < 1$.

1. Introduction and statement of results

For an arbitrary polynomial $f(z)$ let $M(f, r) = \max_{|z|=r} |f(z)|$. Further let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Concerning the estimate of $|p(z)|$ on $|z| \leq r$ we have the following well known result (see [7, Problem III 269, p. 158]).

Theorem 1.1. *If $p(z)$ is a polynomial of degree n then*

$$M(p, R) \leq R^n M(p, 1), R \geq 1,$$

with equality only for $p(z) = \lambda z^n$.

For polynomial not vanishing in $|z| < 1$ Ankeny and Rivlin [1] proved

Theorem 1.2. *Let $p(z)$ be a polynomial of degree n, having no zeros in $|z| < 1$. Then*

$$M(p, R) \leq \frac{R^n + 1}{2} M(p, 1), R \geq 1.$$

The result is the best possible with equality only for the polynomial $p(z) = \lambda + \mu z^n, |\lambda| = |\mu|$.

Dewan and Pukhta [2] used the generalization of Schwarz's lemma [8, p. 212] to obtain the following refinement of Theorem 1.2.

Theorem 1.3. *Let $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$ be a polynomial of degree n and let $|z_t| \geq K_t \geq 1, 1 \leq t \leq n$. Then for $R \geq 1$*

$$\begin{aligned} M(p, R) &\leq \frac{R^n + 1}{2} M(p, 1) - \frac{R^n - 1}{2} B M(p, 1) \\ &- \frac{n}{2} \left\{ \frac{(1 - B)^2 (M(p, 1))^2 - 4|a_n|^2}{(1 - B) M(p, 1)} \right\} \times \Big[\frac{(R - 1)(1 - B) M(p, 1)}{(1 - B) M(p, 1) + 2|a_n|} \\ &- \ln \left\{ 1 + \frac{(R - 1)(1 - B) M(p, 1)}{(1 - B) M(p, 1) + 2|a_n|} \right\} \Big], \end{aligned}$$

where

$$B = \frac{1}{1 + \frac{2}{n} \sum_{t=1}^{n} \frac{1}{K_t - 1}}.$$

In this paper we have used Govil, Rahman and Schmeisser's refinement of the generalization of Schwarz's lemma [4, Lemma] to obtain a new refinement of Theorem 1.2. Our refinement is a refinement of Theorem 1.3 also. More precisely we prove

Theorem 1.4. *Let*

$$p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n} (z - z_t)$$

be a polynomial of degree n such that

$$|z_t| \geq K_t \geq 1, 1 \leq t \leq n.$$

Further let

$$M = \begin{cases} \dfrac{n}{2} \left\{ 1 - \dfrac{1}{1 + \frac{2}{n} \sum\limits_{t=1}^{n} \frac{1}{K_t - 1}} \right\} M(p, 1), & K_t \neq 1 \text{ for all } t \quad (1.1) \\ \dfrac{n}{2} M(p, 1), & K_t = 1 \text{ for certain } t \ (1 \leq t \leq n) \quad (1.2) \end{cases}$$

$$a = n\overline{a_n}, \quad (1.3)$$

$$b = (n - 1)\overline{a_{n-1}} \quad (1.4)$$

$$R \geq 1 \quad (1.5)$$

and

$$D=\begin{cases}\dfrac{1}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\left(\tan^{-1}\dfrac{R+\frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\right.\\ \left.-\tan^{-1}\dfrac{1+\frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\right),\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}>0, & (1.6)\\ \dfrac{1}{2\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\left(\ln\left|\dfrac{R+\frac{|b|}{2(M-|a|)}-\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}{R+\frac{|b|}{2(M-|a|)}+\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\right|-\right.\\ \left.\ln\left|\dfrac{1+\frac{|b|}{2(M-|a|)}-\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}{1+\frac{|b|}{2(M-|a|)}+\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\right|\right),\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}<0, & (1.7)\\ -\left(R+\dfrac{|b|}{2(M-|a|)}\right)^{-1}+\left(1+\dfrac{|b|}{2(M-|a|)}\right)^{-1},\\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}=0. & (1.8)\end{cases}$$

Then

$$M(p,R)\leq$$

$$\begin{cases}\dfrac{R^n+1}{2}M(p,1)-\dfrac{R^n-1}{2}\Big(\dfrac{1}{1+\frac{2}{n}\sum_{t=1}^{n}\frac{1}{K_t-1}}\Big)M(p,1)\\ -(M-|a|)(R-1)+\dfrac{|b|}{2}\ln\dfrac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|}\\ +\dfrac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D,M>|a| \text{ and } \ K_t\neq 1 \text{ for all } t, & (1.9)\\ \dfrac{R^n+1}{2}M(p,1)-(M-|a|)(R-1)\\ +\dfrac{|b|}{2}\ln\dfrac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{M^2-|a|^2+M|b|}\\ +\dfrac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D,\\ M>|a| \text{ and } K_t=1 \text{ for certain } t\ (1\leq t\leq n), & (1.10)\\ \dfrac{R^n+1}{2}M(p,1)-\dfrac{R^n-1}{2}\Big(\dfrac{1}{1+\frac{2}{n}\sum_{t=1}^{n}\frac{1}{K_t-1}}\Big)M(p,1),\\ M=|a| \text{ and } K_t\neq 1\forall t, & (1.11)\\ \dfrac{R^n+1}{2}M(p,1),M=|a| \text{ and } K_t=1 \text{ for certain } t,(1\leq t\leq n). & (1.12)\end{cases}$$

The result is the best possible if $K_t = 1$ for certain $t, (1 \le t \le n)$ and the equality holds for the polynomial $p(z) = \lambda + \mu z^n, |\lambda| = |\mu|$.

Remark 1.5. That Theorem 1.4 is a refinement of Theorem 1.3 can be seen from the fact that a refinement of the generalization of Schwarz's lemma is used to obtain Theorem 1.4.

Further by taking $K_t = K, (K \ge 1), \forall t$, in Theorem 1.4 we get

Corollary 1.6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, having no zeros in $|z| < K, (K \ge 1)$. Further let*

$$\begin{aligned} M &= \frac{n}{1+K} M(p,1), \\ a &= n\overline{a_n}, \\ b &= (n-1)\overline{a_{n-1}}, \\ R &\ge 1 \end{aligned}$$

and

$$D = \begin{cases} \dfrac{1}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} \left(\tan^{-1} \dfrac{R + \frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} - tan^{-1} \dfrac{1 + \frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} \right), \\ \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} > 0, \\ \dfrac{1}{2\sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \left(\ln \left| \dfrac{R + \frac{|b|}{2(M-|a|)} - \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}}{R + \frac{|b|}{2(M-|a|)} + \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \right| - \right. \\ \left. \ln \left| \dfrac{1 + \frac{|b|}{2(M-|a|)} - \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}}{1 + \frac{|b|}{2(M-|a|)} + \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \right| \right), \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} < 0, \\ - \left(R + \dfrac{|b|}{2(M-|a|)} \right)^{-1} + \left(1 + \dfrac{|b|}{2(M-|a|)} \right)^{-1}, \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} = 0. \end{cases}$$

Then

$$M(p,R) \le \begin{cases} \dfrac{R^n + K}{1+K} M(p,1) - (M-|a|)(R-1) \\ + \dfrac{|b|}{2} \ln \dfrac{(M-|a|)MR^2 + M|b|R + |a|(M-|a|)}{(M^2-|a|^2) + M|b|} \\ + \dfrac{2(M-|a|)(M^2-|a|^2) - M|b|^2}{2M(M-|a|)} D, & M > |a|, \\ \dfrac{R^n + K}{1+K} M(p,1), & M = |a|. \end{cases}$$

The result is the best possible if $K = 1$ *and the equality holds for the polynomial* $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.

Remark 1.7. Corollary 1.6 is a refinement of Dewan and Pukhta's result [2, Corollary].

2. Lemmas

For the proof of Theorem 1.4 we require the following lemmas.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree* n*, having no zeros in* $|z| < 1$ *then*

$$M(p', 1) \leq \frac{n}{2} M(p, 1).$$

This lemma is due to Lax [5].

Lemma 2.2. *Let* $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$*, be a polynomial of degree* n*. If* $|z_t| \geq K_t \geq 1$, $1 \leq t \leq n$*, then*

$$M(p', 1) \leq n\Big(\Big(\sum_{t=1}^{n} \frac{1}{K_t - 1}\Big)/\Big(\sum_{t=1}^{n} \frac{K_t + 1}{K_t - 1}\Big)\Big) M(p, 1).$$

The result is the best possible with the equality for the polynomial $p(z) = (z + k)^n$, $k \geq 1$.

This lemma is due to Govil and Labelle [3].

Lemma 2.3. *If* $f(z)$ *is analytic and* $|f(z)| \leq 1$ *in* $|z| < 1$ *then*

$$|f(z)| \leq \frac{(1 - |a'|)|z|^2 + |b'||z| + |a'|(1 - |a'|)}{|a'|(1 - |a'|)|z|^2 + |b'||z| + (1 - |a'|)}, (|z| < 1),$$

where $a' = f(0)$, $b' = f'(0)$*. The example*

$$f(z) = \Big(a' + \frac{b'}{1 + a'} z - z^2\Big)/\Big(1 - \frac{b'}{1 + a'} z - a' z^2\Big)$$

shows that the estimate is sharp.

This lemma is due to Govil et al. [4].

Remark 2.4. By using the result [6, p. 172, exercise # 9] one can show that Lemma 2.3 is a refinement of the generalization of Schwarz's lemma.

Lemma 2.5. *If* $g(z)$ *is analytic in* $|z| \leq 1$*, with*

$$\begin{aligned} |g(z)| &\leq M_1, |z| \leq 1, \\ g(0) &= a_1, \\ g'(0) &= b_1 \end{aligned}$$

then

$$|g(z)| \le \begin{cases} M_1 \dfrac{M_1(M_1-|a_1|)|z|^2 + M_1|b_1||z| + |a_1|(M_1-|a_1|)}{|a_1|(M_1-|a_1|)|z|^2 + M_1|b||z| + M_1(M_1-|a_1|)}, M_1 > |a_1| \text{ and } |z| \le 1, \\ M_1, \quad M_1 = |a_1| \text{ and } |z| \le 1. \end{cases}$$

Proof. It follows easily by applying Lemma 2.3 to the function $g(z)/M_1$. □

3. Proof of Theorem 1.4

For the polynomial

$$T(z) = z^{n-1}\overline{p'(1/\overline{z})} \tag{3.1}$$

we have

$$|T(z)| = |p'(z)|, |z| = 1,$$

which by Lemma 2.1, Lemma 2.2, (1.1) and (1.2) implies that

$$|T(z)| \le M, |z| \le 1.$$

Therefore on applying Lemma 2.5 to $T(z)$ we get for $|z| \le 1$

$$|T(z)| \le \begin{cases} M \dfrac{M(M-|a|)|z|^2 + M|b||z| + |a|(M-|a|)}{|a|(M-|a|)|z|^2 + M|b||z| + M(M-|a|)}, \ M > |a|, (\text{by (1.3) and (1.4)}), \\ M, \quad M = |a|, \ (\text{by (1.3)}), \end{cases}$$

which on using (3.1) and

$$z = \frac{1}{R}e^{i\theta}, \qquad 0 \le \theta \le 2\pi,$$

implies for $0 \le \theta \le 2\pi$

$$|p'(Re^{i\theta})| \le \begin{cases} MR^{n-1}\left\{1 - \dfrac{(M-|a|)^2(R^2-1)}{|a|(M-|a|) + M|b|R + M(M-|a|)R^2}\right\}, \\ M > |a|, (\text{by (1.5)}) & (3.2) \\ MR^{n-1}, \quad M = |a|, \ (\text{by (1.5)}). & (3.3) \end{cases}$$

Now we consider the case $M > |a|$. For $0 \le \theta \le 2\pi$ we have

$$
\begin{aligned}
|p(Re^{i\theta}) - p(e^{i\theta})| &= |\int_1^R p'(re^{i\theta})e^{i\theta}dr| \text{ (by (1.5))} \\
&\leq \int_1^R |p'(re^{i\theta})|dr \leq M\int_1^R r^{n-1}dr \\
&- M(M-|a|)^2\int_1^R \frac{r^{n-1}(r^2-1)}{|a|(M-|a|)+M|b|r+M(M-|a|)r^2}dr \text{ (by (3.2))} \\
&\leq M\frac{R^n-1}{n} - M(M-|a|)^2\int_1^R \frac{r^2-1}{|a|(M-|a|)+M|b|r+M(M-|a|)r^2}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)\int_1^R dr \\
&+ (M-|a|)\int_1^R \frac{M|b|r+M^2-|a|^2}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) \\
&+ \frac{|b|}{2}\int_1^R \frac{2M(M-|a|)r+M|b|}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&+ \frac{1}{2}\times\int_1^R \frac{2(M^2-|a|^2)(M-|a|)-M|b|^2}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) \\
&+ \frac{|b|}{2}\ln\frac{M(M-|a|)R^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} \\
&+ \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}\int_1^R \frac{1}{\left\{r+\frac{|b|}{2(M-|a|)}\right\}^2+\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) + \\
&\quad \frac{|b|}{2}\ln\frac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} + \\
&\quad \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D \text{ (by (1.6), (1.7) and (1.8))},
\end{aligned}
$$

which implies

$$
\begin{aligned}
M(p,R) \leq\; & M(p,1) + M\frac{R^n-1}{n} - (M-|a|)(R-1) + \\
& \frac{|b|}{2}\ln\frac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} + \\
& \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D
\end{aligned}
$$

and inequalities (1.9) and (1.10) follow respectively by using relations (1.1) and (1.2).

Further we consider the possibility $M = |a|$. The proof of inequalities (1.11) and (1.12) is similar to the proof of inequalities (1.9) and (1.10), with one change:

inequality (3.3) instead of inequality (3.2)

and so we omit the details. This completes the proof of Theorem 1.4.

References

[1] N.C. Ankeny, T.J. Rivlin, *On a theorem of S. Bernstein*, Pac. J. Math. 5 (1955) 849–852.

[2] K.K. Dewan, M.S. Pukhta, *On the maximum modulus of polynomials*, BHKMS 2 (1999) 279–286.

[3] N.K. Govil, G. Labelle, *On Bernstein's inequality*, J. Math. Anal. Appl. 126 (1987) 494–500.

[4] N.K. Govil, Q.I. Rahman, G. Schmeisser, *On the derivative of a polynomial*, Ill. J. Maths. 23 (1979) 319–329.

[5] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. 50 (1944) 509–513.

[6] Z. Nehari, *Conformal Mapping*, 1^{st} ed., McGraw-Hill, New York, 1952.

[7] G. Polya, G. Szegö, *Problems and Theorems in Analysis, Vol.* 1, Springer-Verlag, Berlin-Heidelberg, 1972.

[8] E.C. Titchmarsh, *The Theory of Functions*, The English Language Book Society and Oxford University Press, London, 1962.

DOI: 10.7862/rf.2019.7

Vinay Kumar Jain
email: vinayjain.kgp@gmail.com
ORCID: 0000-0003-2382-2499
Mathematics Department
I.I.T.
Kharagpur - 721302
INDIA

Received 18.09.2018 *Accepted 10.01.2019*