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Involutory BE–algebras

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Abstract: This paper is devoted to the study of some structural properties of bounded and involutory BE–algebras and investigate the relationship between them. We construct a commutative monoid by definition of proper operation in an involutory BE–algebra. Some rules of calculus for BE–algebras with a semi-lattice structure are provided. Many results related to the natural order of a BE–algebras were found. Finally, we show that an involutory bounded BE–algebra X is semi-simple.

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1 Introduction and Preliminaries

The study of BCK/BCI–algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic. There exist several generalization of BCK/BCI–algebras, such as BCH–algebras, d–algebras, B–algebras, BH–algebras, etc.

Especially, the notion of BE–algebras was introduced by H. S. Kim and Y. H. Kim [7], in which was deeply studied by S. S. Ahn and et. al., in [1, 2, 3], Walendziak in [15], A. Rezaei and et. al., in [12, 13, 14]. Lattice-valued logic is becoming a research filed strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. BE–algebras are important tools for certain investigations in algebraic logic since they can be consider as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "*true*". In this paper, we develop the theory BE–algebras with define a new structure as bounded and involutory BE–algebras and investigate the relationship between them and proved some theorems.

The paper has been organized in tree sections. In section 1, we give some definitions and some previous results and in section 2 we define bounded BE–algebras and define a congruence relation on this algebra with respect to a filter which this congruence relation allowed us to define a quotient algebra is also a bounded BE–algebra. In section 3 we discus on involutory BE–algebra because it is well known this structure has an important and vital role in investigating the structure of a logical system. Since quotient algebra is a basic tool for exploring the structures of algebras and there are close contacts among congruences and quotient algebras, we introduce a new congruence relation on X and construct quotient algebra via this congruence relation.

Definition 1.1. *[7] An algebra $(X;*,1)$ of type $(2,0)$ is called a BE–algebra if following axioms hold:*

$(BE1)$ $x*x=1$,

$(BE2)$ $x*1=1$,

$(BE3)$ $1*x=x$,

$(BE4)$ $x*(y*z)=y*(x*z)$, *for all* $x,y,z\in X$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$.

Proposition 1.2. *[7] Let X be a BE–algebra. Then*

(i) $x*(y*x)=1$,

(ii) $y*((y*x)*x)=1$, *for all* $x,y\in X$.

From now on, in this paper X is a BE–algebra, unless otherwise is stated. A subset F of X is called a filter of X if $(F1)$ $1\in F$ and $(F2)$ $x\in F$ and $x*y\in F$ imply $y\in F$. We denote By $F(X)$ the set of all filters of X and $Max(X)$ the set of all maximal filters of X. Let A be a non-empty subset of X, then the set

$$< A >= \bigcap\{G\in F(X)|\ A\subseteq G\}$$

is called the filter generated by A, written $< A >$. If $A=\{a\}$, we will denote $<\{a\}>$, briefly by $< a >$, and we call it a principal filter of X. For $F\in F(X)$ and $a\in X$, we denote by F_a the filter generated by $F\cup\{a\}$. X is said to be self distributive if $x*(y*z)=(x*y)*(x*z)$, for all $x,y,z\in X$, (Example 8., [7]).

In a self distributive BE–algebra X, $F_a=\{x\in X: a*x\in F\}$,([3]). X is said to be transitive if $y*z\leq(x*y)*(x*z)$ for all $x,y,z\in X$, [1]. We say that X is commutative if $(x*y)*y=(y*x)*x$, for all $x,y\in X$. In [15], A. Walendziak, showed that every dual BCK–algebra is a BE–algebra and any commutative BE–algebra is a dual BCK–algebra.

We note that "$\leq$" is reflexive by $(BE1)$. If X is self distributive, then relation "$\leq$" is a transitive order set on X. Because if $x\leq y$ and $y\leq z$, then

$$x*z=1*(x*z)=(x*y)*(x*z)=x*(y*z)=x*1=1$$

and so $x \leq z$. If X is commutative, then by Proposition 3.3, [15], relation "$\leq$" is antisymmetric. Hence if X is a commutative self distributive BE–algebra, then "$\leq$" is a partial order set on X, (Example 3.4., [3]). We show that if I be an obstinate ideal of a self distributive BE–algebra X, then $(X/I; *, C_1)$ is also a BE-algebra, which is called to be the quotient algebra via I, and $C_1 = I$, (see Theorem 3.13, [12]).

Proposition 1.3. *[12] Let X be self distributive. If $x \leq y$, then*

- (i) $z * x \leq z * y$ *and* $y * z \leq x * z$,
- (ii) $y * z \leq (z * x) * (y * x)$, *for all* $x, y, z \in X$.

Theorem 1.4. *[13] A dual BCK–algebra X is commutative if and only if $(X; \leq)$ is an upper semi-lattice with $x \vee y = (y * x) * x$, for all $x, y \in X$.*

Proposition 1.5. *[13] Let X be a commutative BE–algebra. Then*

- (i) *for each $a \in X$, the mapping $f_a : x \to x * a$ is an anti-tone involution on the section $[a, 1]$.*
- (ii) *$(A, \leq)$ is a near-lattice with section anti-tone involutions and for every $a \in X$, the anti-tone involutions f_a on $[a, 1]$ is given by $f_a(x) = x * a$.*

Theorem 1.6. *[15, 13] Let X be commutative. Then it is a semi-lattice with respect to $\vee$.*

Definition 1.7. *[4] A filter F of X is called an obstinate filter if $x, y \notin F$ imply $x * y \in F$ and $y * x \in F$.*

Theorem 1.8. *[5] Let X be self distributive. $F \in F(X)$ and $F \neq X$. Then the following are equivalent:*

- (i) *F is an obstinate filter,*
- (ii) *if $x \notin F$, then $x * y \in F$, for all $y \in F$.*

2 On Bounded BE-algebras

Definition 2.1. *X is called bounded if there exists the smallest element 0 of X (i.e., $0 * x = 1$, for all $x \in X$).*

Example 2.2. (i). *The interval $[0, 1]$ of real numbers with the operation "$*$" defined by*

$$x * y = \min\{1 - x + y, 1\}, \ \ for\ all\ x, y \in X$$

is a bounded BE–algebra.

(ii). *Let $(X; *, 1)$ be a BE–algebra, $0 \notin X$ and $\bar{X} = X \cup \{0\}$. If we extensively define*

$$0 * x = 0 * 0 = 1 \ \ and \ \ x * 0 = 0 \ \ for\ all\ x \in X.$$

Then $(\bar{X};*,0,1)$ *is a bounded* BE*–algebra with* 0 *as the smallest element.*
(iii). *Let* $X := \{0,a,b,1\}$ *be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	a	1	b	1
b	b	a	1	1
1	0	a	b	1

Then $(X;*,0,1)$ *is a bounded* BE*–algebra with* 0 *as the smallest element.*
(iv). *Let* $X := \{0,a,b,c,1\}$ *be a set with the following table.*

$*$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	a	1	c	1
c	0	1	b	1	1
1	0	a	b	c	1

Then $(X;*,0,1)$ *is a bounded* BE*–algebra with* 0 *as the smallest element.*
(v). *Let* $(X;\vee,\wedge,\neg,0,1)$ *be a Boolean-lattice. Then* $(X;*,1)$ *is a bounded* BE*–algebra, where operation "* $*$ *" is defined by* $x*y=(\neg x)\vee y$*, for all* $x,y\in X$.

Remark. The following example shows that the bounded BE–algebra is not a dual BCK–algebra and Hilbert algebra in general (see Definition 2.3, [15] and Definition 3.1, [14]).

Example 2.3. *Let* $X := \{0,a,b,1\}$ *be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	a	b	1

Then $(X;*,0,1)$ *is a bounded* BE*–algebra with* 0 *as the smallest element but it is not a dual* BCK*–algebra, Hilbert algebra. Because*

$$a*b=b*a=1 \quad \textit{while} \quad a\neq b.$$

Also, it is not an implication algebra. Because

$$(a*b)*b=1*b=b\neq(b*a)*a=1*a=a.$$

Given a bounded BE–algebra X with 0 as the smallest element, we denote $x*0$ by Nx, then N can be regarded as a unary operation on X.

Proposition 2.4. *Let* X *be bounded with the smallest element* 0. *Then the following hold:*

(*i*) $N0 = 1$ *and* $N1 = 0$,

(*ii*) $x \leq NNx$,

(*iii*) $x * Ny = y * Nx$, *for all* $x, y \in X$.

Proof. (*i*). By $(BE1)$ and $(BE2)$ we have $N0 = 0 * 0 = 1$ and $N1 = 1 * 0 = 0$.
(*ii*). Since $x * (Nxx) = x * ((x * 0) * 0) = (x * 0) * (x * 0) = 1$, then $x \leq NNx$.
(*iii*). By $(BE4)$ we have $x * Ny = x * (y * 0) = y * (x * 0) = y * Nx$. □

Proposition 2.5. *Let X be a self distributive and bounded. Then*

(*i*) $y * x \leq Nx * Ny$,

(*ii*) $x \leq y$, *implies* $Ny \leq Nx$, *for all* $x, y \in X$.

Proof. (*i*). We have

$$\begin{aligned} (y * x) * (Nx * Ny) = Nx * ((y * x) * Ny) &= (x * 0) * ((y * x) * (y * 0)) \\ &= (x * 0) * (y * (x * 0)) \\ &= y * ((x * 0) * (x * 0)) \\ &= y * 1 = 1. \end{aligned}$$

Hence $y * x \leq Nx * Ny$.
(*ii*). By $(BE3)$ and assumption we have

$$\begin{aligned} Ny * Nx = (y * 0) * (x * 0) &= (y * 0) * (1 * (x * 0)) \\ &= (y * 0) * ((x * y) * (x * 0)) \\ &= (y * 0) * (x * (y * 0)) \\ &= x * ((y * 0) * (y * 0)) \\ &= x * 1 = 1. \end{aligned}$$

Hence $Ny \leq Nx$. □

In the following example we show that the self-distributivity condition in the above theorem is necessary.

Example 2.6. *Example 2.2(iii), is a bounded BE–algebra with* 0 *as the smallest element, while it is not self-distributive. Because*

$$b * (0 * a) = 2 * 1 = 1 \neq (b * 0) * (b * a) = b * a = a.$$

We can seen easily that, $b = a * b \nleq Nb * Na = b * a = a$.

Proposition 2.7. *Let X be bounded implicative self distributive. Then the following hold:*

(*i*) *X is commutative,*

(*ii*) $x = Nx * x$,

(*iii*) $x \vee y = y \vee x = Nx * y$, *for all* $x, y \in X$.

Proof. (*i*). See proof of Theorem 3.12([14]).

(*ii*). Assume that X is a bounded implicative. Then $Nx * x = (x * 0) * x = x$.

(*iii*). Let X be bounded implicative self distributive and $x, y \in X$. then by Proposition 1.3, $0 \leq y$ and $x * 0 \leq x * y$. Furthermore, by Propositions 1.2 and 1.3, we get

$$x \leq (x * y) * y \leq (x * 0) * y = Nx * y$$

Since by Proposition 1.2, $y \leq Nx * y$, then $Nx * y$ is an upper bound of x and y. Hence $x \vee y \leq Nx * y$. Also, we have

$$Nx * y \leq (y * x) * (Nx * x) = (y * x) * x.$$

Since X is commutative, then by Theorem 1.6, we have $(y * x) * x = x \vee y = y \vee x$ and so by Proposition 3.3([15]), the proof is complete. □

Corollary 2.8. *Let* X *be self distributive,* $F \in F(X)$ *and* $F \neq X$*. Then the following are equivalent:*

(*i*) F *is an obstinate filter,*

(*ii*) *if* $x \notin F$*, then* $Nx \in F$*.*

Definition 2.9. *Let* X *and* Y *be bounded. A homomorphism from* X *to* Y *is a function* $f : X \to Y$ *such that*

(*i*) $f(x * y) = f(x) * f(y)$,

(*ii*) $f(Nx) = N(f(x))$,

(*iii*) $f(0) = 0$, *for all* $x, y \in X$.

Example 2.10. *Consider* X *as Example 2.2(iii) and* Y *as Example 2.3. Define* $f : X \to Y$ *such that* $f(1) = f(a) = f(b) = 1$ *and* $f(0) = 0$*. Then* f *is a homomorphism.*

Theorem 2.11. *Let* $f : X \to Y$ *be a homomorphism. Then* $\ker(f) = \{x \in X : f(x) = 1\}$ *is a filter in* X*. Moreover, if* $f(x) = f(y)$*, then* $x * y \in \ker(f)$ *and* $y * x \in \ker(f)$*, for all* $x, y \in X$*. If* Y *is commutative, then the converse is valid.*

Proof. We have $f(1) = f(x * x) = f(x) * f(x) = 1$. Hence $1 \in \ker(f)$. Now, let $x \in \ker(f)$ and $x*y \in \ker(f)$. Then $f(x) = f(x*y) = 1$. But $f(x*y) = f(x)*f(y) = 1$. Hence $f(y) = 1 * f(y) = 1$. Therefore, $y \in \ker(f)$.

Now, let $f(x) = f(y)$. By using $(BE1)$, $f(x) * f(y) = 1$ and $f(y) * f(x) = 1$. But $1 = f(x) * f(y) = f(x * y)$ and $1 = f(y) * f(x) = f(y * x)$ implies $x * y \in \ker(f)$ and $y * x \in \ker(f)$.

Assume that Y is commutative, $x * y \in \ker(f)$ and $y * x \in \ker(f)$. Then $f(x * y) = f(y * x) = 1$ which implies that $f(x) * f(y) = f(y) * f(x) = 1$. Hence by Proposition 3.3([15]), $f(x) = f(y)$. □

Theorem 2.12. *Let X be bounded transitive, F be a filter and X/F be the corresponding quotient algebra. Then the map $f : X \to X/F$ which is defined by $f(a) = [a]$, for all $a \in X$, is a homomorphism and* $\ker(f) = F$.

Proof. By Propositions 5.4 and 5.7([11]), X/F is a quotient BE–algebra. Now, we have $f(0) = [0]$ and

$$f(Nx) = f(x * 0) = f(x) * f(0) = f(x) * [0] = N(f(x)).$$

Now, let $x \in \ker(f)$. Then $f(x) = [x] = [1]$ if and only if $1 = x * 1 \in F$ and $x = 1 * x \in F$ if and only if $1 \in F$ and $x \in F$. Therefore, $\ker(f) = F$. □

3 Involutory BE–algebras

If $NNx = x$, then x is called an involution of X. The smallest element 0 and the greatest element 1 are two involutions of X, because

$$NN0 = N(0 * 0) = N1 = 1 * 0 = 0,$$

$$NN1 = N(1 * 0) = N0 = 0 * 0 = 1.$$

Definition 3.1. *A bounded BE–algebra X is called involutory if any element of X is involution.*

Example 3.2. (i). *Examples 2.2*(i), (iii), (v), *are involutory.*
(ii). *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

Then $(X; *, 0, 1)$ is a bounded BE–algebra but it is not an involutory. Because

$$NNb = N(b * 0) = N0 = 0 * 0 = 1 \neq b.$$

(iii). *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	1	1	1
1	0	a	b	1

Then $(X; *, 0, 1)$ is an involutory BE–algebra but it is not an involutory dual BCK–algebra and involutory Hilbert algebra. Because

$$a * b = 1 \text{ and } b * a = 1 \text{ while, } a \neq b.$$

Also, it is not an involutory implication algebra. Because

$$(a * b) * b = 1 * b = b \neq (b * a) * a = 1 * a = a.$$

Proposition 3.3. *If X is a bounded commutative, then X is an involutory.*

Proof. By using the commutativity we get

$$NNx = (x * 0) * 0 = (0 * x) * x = 1 * x = x.$$

Hence X is an involutory. □

In the following example we show that the commutativity condition in the above theorem is necessary.

Example 3.4. *Example 3.2(ii), is not an involutory. Because it is not commutative.*

Proposition 3.5. *If X is an involutory, then*

(i) $x * y = Ny * Nx$,

(ii) $x \leq Ny$ *implies* $y \leq Nx$, *for all* $x, y \in X$.

Proof. (i). Since X is an involutory, then we have $NNx = x$, for all $x, y \in X$. Hence by Proposition 2.4(iii), $x * y = x * NNy = Ny * Nx$.

(ii). Since $x \leq Ny$, we get $x * Ny = 1$. Hence by Proposition 2.4(iii), $1 = x * Ny = y * Nx$. So, $y \leq Nx$. □

Lemma 3.6. *Let X be bounded self distributive and $x, y \in X$.*

(i) *if the smallest upper bound $x \vee y$ of x and y exists, then the greatest lower bound $Nx \wedge Ny$ of Nx and Ny exists and $Nx \wedge Ny = N(x \vee y)$.*

(ii) *if X is involutory and the greatest lower bound $x \wedge y$ exists, then the least upper bound $Nx \vee Ny$ exists and $Nx \vee Ny = N(x \wedge y)$.*

Proof. (i). Assume that the smallest upper bound $x \vee y$ of x and y exists. Since $x \leq x \vee y$, then by Proposition 1.3, $(x \vee y) * 0 \leq x * 0$, (i.e., $N(x \vee y) \leq Nx$). By the similar way $N(x \vee y) \leq Ny$. Hence $N(x \vee y)$ is a lower bound of Nx and Ny. Also, assume that u is any lower bound of Nx and Ny. Then $u \leq Nx$ and $u \leq Ny$. Hence by $(BE4)$, we have $x * (u * 0) = u * (x * 0) = u * Nx = 1$. Hence $x \leq Nu$ and by the similar way $y \leq Nu$. So, $x \vee y \leq Nu$. Now, by $(BE4)$, we have $(x \vee y) * (u * 0) = u * ((x \vee y) * 0) = 1$. So, $u \leq N(x \vee y)$. Hence $N(x \vee y)$ is a greatest lower bound of Nx and Ny. Therefore, the greatest lower bound $Nx \wedge Ny$ of Nx and Ny exists, and $Nx \wedge Ny = N(x \vee y)$.

(ii). Assume that $x \wedge y$ exists. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, then by Proposition 2.5, we have $N(x) \leq N(x \wedge y)$ and $N(y) \leq N(x \wedge y)$. Hence $N(x \wedge y)$ is an upper bound of Nx and Ny. Also, let u be any upper bound of Nx and Ny. Then $Nx \leq u$ and $Ny \leq u$. Since X is involutory, then by Proposition 2.5, we derive $Nu \leq NNx = x$ and $Nu \leq NNy = y$. So, $Nu \leq x \wedge y$. By Proposition 2.5, we have $N(x \wedge y) \leq NNu = u$. Hence $N(x \wedge y)$ is the smallest upper bound of Nx and Ny. Then the least upper bound $Nx \vee Ny$ exists, and $Nx \vee Ny = N(x \wedge y)$. □

Theorem 3.7. *Let X be involutory self distributive. Then the following are equivalent:*

(i) $(X;\leq)$ *is an upper semi-lattice,*

(ii) $(X;\leq)$ *is a lower semi-lattice,*

(iii) $(X;\leq)$ *is a lattice.*

Moreover, if $(X;\leq)$ is a lattice, then the following identities hold:

$$x\wedge y = N(Nx\vee Ny) \text{ and } x\vee y = N(Nx\wedge Ny).$$

Proof. $(i)\Rightarrow(ii)$. Since $(X;\leq)$ is an upper semi-lattice, then $Nx\vee Ny$ exists for all $x,y\in X$. By the first half part of Lemma 3.6, $NNx\wedge NNy$ exists. Also, since X is involutory, we have $NNx\wedge NNy = x\wedge y$. Then $x\wedge y$ exists. So, $(X;\leq)$ is a lower semi-lattice.

$(ii)\Rightarrow(iii)$. Since $(X;\leq)$ is a lower semi-lattice, $Nx\wedge Ny$ exists and using the second half part of Lemma 3.6, $NNx\vee NNy$ exists, for all $x,y\in X$. Also, since X is involutory, we have $NNx\vee NNy = x\vee y$. Then $x\vee y$ exists. So, $(X;\leq)$ is an upper semi-lattice.

$(iii)\Rightarrow(i)$. The proof is obvious.

Now, let $(X;\leq)$ is a lattice. Since by Lemma 3.6, X is involutory, then we have

$$x\wedge y = NNx\wedge NNy = N(Nx\vee Ny),$$

$$x\vee y = NNx\vee NNy = N(Nx\wedge Ny).$$

□

Theorem 3.8. *Let I be an obstinate ideal of involutory(bounded) self-distributive X. Then $(X/I;*,C_1)$ is involutory(bounded) self-distributive, too.*

Proof. By Theorems 3.13 and 3.16([12]), $(X/I;*,C_1)$ is a self-distributive BE–algebra. Let $x\in X$. Then $C_0*C_x = C_{0*x} = C_1$. Hence X/I is a bounded BE–algebra. Now,

$$NNC_x = (C_x*C_0)*C_0 = C_{x*0}*C_0 = C_{(x*0)*0} = C_{NNx} = C_x.$$

Therefore, X/I is an involutory BE–algebra. □

Proposition 3.9. *Let X be involutory and operation "$\circ$" is defined on X by $x\circ y = Nx*y$, for all $x,y\in X$. Then $(X;\circ,0)$ is a commutative monoid.*

Proof. By Proposition 2.4(iii),

$$x\circ y = Nx*y = Nx*NNy = Ny*NNx = Ny*x = y\circ x$$

and so X is commutative. Now, by Proposition 2.4(iii), and ($BE4$) we have

$$\begin{aligned} x \circ (y \circ z) = Nx * (y \circ z) &= Nx * (z \circ y) \\ &= Nx * (Nz * y) \\ &= Nz * (Nx * y) \\ &= z \circ (Nx * y) \\ &= (Nx * y) \circ z \\ &= (x \circ y) \circ z. \end{aligned}$$

Hence " $\circ$ " is associative operation on X. Moreover, for any $x \in X$

$$x \circ 0 = Nx * 0 = NNx = x \ \ and \ \ 0 \circ x = N0 * x = 1 * x = x.$$

□

In the following example we show that the converse of the Proposition 3.9, is not valid in general.

Example 3.10. *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	1	a
1	1	a
a	a	a

Then $(X; *, 1)$ is a commutative monoid, but it is not a BE–algebra. Because $a * a = a \neq 1$ and $a * 1 = a \neq 1$, (i.e., conditions ($BE1$) and ($BE2$) are not holds).

Lemma 3.11. *Let X be bounded. Then*

(i) *filter F of X is proper if and only if $0 \notin F$.*

(ii) *each proper filter F is contained in a maximal filter.*

Proof. (i). Let F be a proper filter of X and $0 \in F$. If $x \in X$, since $0 * x = 1 \in F$, which implies $x \in F$. Hence $X = F$, which is a contradiction. The converse is clear. (ii). The proof is obvious. □

Theorem 3.12. *Every bounded BE–algebra contains at least one maximal filter.*

Proof. Let X be a bounded BE–algebra. Since $\{1\}$ is a proper filter of X, then the proof is clear by Lemma 3.11. □

Definition 3.13. *Let X be bounded. Then the radical of X, written $Rad(X)$, is defined by*

$$Rad(X) = \cap\{F : \ F \in Max(X)\}.$$

In view of Theorem 3.12, $Rad(X)$ always exists for a bounded algebra X. Following a standard terminology in the contemporary algebra, we shall call an algebra X semi-simple if $Rad(X) = \{1\}$.

Example 3.14. *In Example 2.2(iv), $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, a, b, c\}$ and X are filters in X and F_3 is only maximal filter of X. Hence $Rad(X) = F_3$.*

Example 3.15. *In Example 2.2(iii), $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$ and X are filters in X and F_2, F_3 are maximal filters of X, also $F_2 \cap F_3 = \{1\}$. Hence $Rad(X) = \{1\}$ and therefore X is semi-simple.*

Lemma 3.16. *Let X be an involutory bounded BE–algebra. Then for every $x \in X$ with $x \neq 1$, there exists a maximal filter F of X such that $x \notin F$.*

Proof. Let $1 \neq x \in X$. We claim that $< Nx >$ is a proper filter of X. By contrary, if it is not, then $< Nx >= X$. Hence $0 \in< Nx >$ and therefore $Nx * 0 = NNx = 1$. Since X is involutory, then $x = NNx = 1$, which is a contradiction. By Lemma 3.11(ii), there is a maximal filter F of X such that $< Nx >\subseteq F$, and $x \notin F$. Suppose $x \in F$. Since $Nx = x * 0 \in F$, then $0 \in F$, which is contrary by Lemma 3.11(i). □

Theorem 3.17. *Let X be involutory and bounded. Then X is a semi-simple.*

Proof. By Lemma 3.16, the proof is clear. □

In this section we define a congruence relation "θ" on involutory bounded BE–algebra X and construct quotient algebra $(X/\theta; *, \theta_0, \theta_1)$ induced by the congruence relation "θ", where, we denote θ_x for the equivalence class $[x]$ containing x. Since "θ" is a congruence on X, then the operation " $*$ " on X/θ given by $\theta_x * \theta_y = \theta_{x*y}$ is well-defined, because "θ" satisfied of the substitution property. Then $(X/\theta; *, \theta_0, \theta_1)$ is an algebra of type $(2, 0, 0)$ where,

$$\theta_0 = \{x : N0 = Nx\} = \{x : Nx = 1\}$$

is the zero equivalence class containing 0 and

$$\theta_1 = \{x : N1 = Nx\} = \{x : Nx = 0\}$$

is the one equivalence class containing 1. Now, in the following theorem define and prove this results.

Theorem 3.18. *Let X be involutory and bounded. The relation "θ" defined on X by:*

$$(x, y) \in \theta \quad if \ and \ only \ if \quad Nx = Ny$$

is a congruence relation on X and the quotient algebra $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra.

Proof. It is clear that "θ" is an equivalence relation on X. Now, Let $(x, y) \in \theta$ and $(u, v) \in \theta$. Then $Nx = Ny$ and $Nu = Nv$. Hence,

$$Nx * Nu = (x * 0) * (u * 0) = u * ((x * 0) * 0) = u * NNx = u * x.$$

Thus $N(u * x) = N(Nx * Nu) = N(Ny * Nv) = N(v * y)$, and so $(u * x, v * y) \in \theta$. By the similarly way we have $(x * u, y * v) \in \theta$. Hence "$\theta$" is a congruence relation on X.

Let $\theta_x, \theta_y, \theta_z \in X/\theta$. Then
$(BE1)$ $\theta_x * \theta_x = \theta_{x*x} = \theta_1$,
$(BE2)$ $\theta_x * \theta_1 = \theta_{x*1} = \theta_1$,
$(BE3)$ $\theta_1 * \theta_x = \theta_{1*x} = \theta_x$,
$(BE4)$ $\theta_x * (\theta_y * \theta_z) = \theta_x * \theta_{y*z} = \theta_{x*(y*z)} = \theta_{y*(x*z)} = \theta_y * \theta_{x*z} = \theta_y * (\theta_x * \theta_z)$.

Now, since $\theta_0 * \theta_x = \theta_{0*x} = \theta_1$. Hence θ_0 is as the smallest element of X/θ. Also,

$$NN\theta_x = (\theta_x * \theta_0) * \theta_0 = \theta_{x*0} * \theta_0 = \theta_{(x*0)*0} = \theta_{NNx} = \theta_x.$$

Therefore, $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra. $\square$

Example 3.19. *Consider Example 2.2(iv), $\theta_0 = \{0\}$ and $\theta_a = \theta_b = \theta_c = \theta_1 = \{a, b, c, 1\}$. Then $X/\theta = \{\theta_0, \theta_1\}$. Thus $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra.*

Corollary 3.20. *Let X be an involutory bounded BE–algebra and $X_0 := \{Nx : x \in X\}$. Then $(X_0; *, N0)$ is a BE–algebra.*

Example 3.21. *In Example 2.2(ii), (iv), respectively, $X_0 = \{0, 1\}$ and $X_0 = \{0, a, b, 1\}$.*

Proposition 3.22. *Let X be involutory, bounded and self-distributive(commutative). Then X/θ is involutory, bounded and self-distributive(commutative), too.*

4 Conclusion and future research

In this paper, we introduced the notion of bounded and involutory BE–algebras and get some results. In addition, we have defined a congruence relation on involutory bounded BE–algebras and construct the quotient BE–algebra via this relations. In [10], J. Meng proved that implication algebras are dual to implicative BCK–algebras. Also R. Halaŝ in [9], showed commutative *Hilbert* algebras are implication algebras and A. Digo in [6], proved implication algebras are *Hilbert* algebras. Recently, A. Walendziak in [15], showed that an implication algebra is a BE–algebra and commutative BE–algebras are dual BCK–algebras. In [14], we showed that every *Hilbert* algebra is a self distributive BE–algebra and commutative self distributive BE–algebra is a *Hilbert* algebra. Then in the following diagram we summarize the results of this paper and we give the relations among such structures of involutory algebras.

"$A \to B$," means that A conclude B.

$$\begin{tikzcd}
 & \text{invo-}BE\text{–algebra} \arrow[dl, shift left, "\text{com}"'] \arrow[d, shift left] \arrow[dr, shift left] & \\
\text{invo-dual } BCK\text{–algebra} \arrow[ur, shift left] \arrow[r, shift left, "\text{imp}"] & \text{invo-implication algebra} \arrow[l, shift left] \arrow[u, shift left, "\text{imp-com}"] \arrow[r, shift left] & \text{invo- } Hilbert\text{ algebra} \arrow[l, shift left, "\text{com}"] \arrow[ul, shift left, "\text{self-com}"']
\end{tikzcd}$$

We think such results are very useful for study in this structure. In the future work we try assemble of calculus relative to different kinds of BE–algebras, as example, latticeal structure and Boolean lattices.

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