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The Real and Complex Convexity

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ABSTRACT: We prove that the holomorphic differential equation $\varphi''(\varphi+c) = \gamma(\varphi')^2$ ($\varphi : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $(\gamma, c) \in \mathbb{C}^2$) plays a classical role on many problems of real and complex convexity. The condition exactly $\gamma \in \{1, \frac{s-1}{s}/s \in \mathbb{N}\backslash\{0\}\}$ (independently of the constant c) is of great importance in this paper.

On the other hand, let $n \geq 1$, $(A_1, A_2) \in \mathbb{C}^2$, and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions. Put $u(z,w) =\mid A_1w - g_1(z) \mid^2 + \mid A_2w - g_2(z) \mid^2$, $v(z,w) =\mid A_1w - \overline{g_1}(z) \mid^2 + \mid A_2w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We prove that u is strictly plurisubharmonic and convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$ and the functions g_1 and g_2 have a classical representation form described in the present paper.

Now v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, $n \in \{1, 2\}$ and g_1, g_2 have several representations investigated in this paper.

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1. Introduction

It is not difficult to prove that if $g : D \to \mathbb{C}$ be a function (not necessarily holomorphic) such that v is convex over $D \times \mathbb{C}$, then g is an affine function, where D is a convex domain of $\mathbb{C}^n$, $n \geq 1$ and $v(z,w) =\mid w - g(z) \mid^2$, for $(z,w) \in D \times \mathbb{C}$.
But if we consider the case of 2 functions, the problem is difficult. However if $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be 2 holomorphic functions, $v_1(z,w) =\mid A_1w - g_1(z) \mid^2$ $+ \mid A_2w - g_2(z) \mid^2$, $v_2(z,w) = v_1(\overline{z}, w)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $A_1, A_2 \in \mathbb{C}$.

We have the questions:

– Find exactly all the conditions described by g_1 and g_2 such that v_1 is convex over $\mathbb{C}^n \times \mathbb{C}$?

– Find exactly all the conditions described by g_1 and g_2 such that v_1 (respectively v_2) is convex and not strictly psh over $\mathbb{C}^n \times \mathbb{C}$?

– Find exactly all the conditions described by g_1 and g_2 such that v_1 (respectively v_2) is convex and strictly psh over $\mathbb{C}^n \times \mathbb{C}$?

Several questions can be studied in this situation.

The class of convex and strictly psh functions is a good family for the study and has several applications in complex analysis, convex analysis in several complex variables, harmonic analysis (representation theory), physics, mechanics and others. For example, the importance of my study of this last class is to discover the existence of an infinite family of convex and strictly psh functions but not strictly convex (or not strictly convex in all Euclidean open ball of the domain of definition) on the above form. It follows that the exact characterization of the (convex and strictly psh) functions of the form $\mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - g_2(z) \mid^2$ describe the existence of an important family of holomorphic functions (which is fundamental for the study). Note that if n increases, the problem is difficult if we consider several absolute values.

Using this paper, we can answer to the following question.
Characterize all the holomorphic not constant functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ and all the holomorphic not constant functions $F_1, F_2 : \mathbb{C}^m \to \mathbb{C}$, such that u is convex (respectively convex and strictly psh) over $\mathbb{C}^n \times \mathbb{C}^m$, where $n, m \geq 1$ and

$$u(z,w) = \mid f_1(z) - F_1(w) \mid^2 + \mid f_2(z) - F_2(w) \mid^2$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

Now, for example, given $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$ and $A_1, A_2 \in \mathbb{C}\backslash\{0\}$. Define $u(z,w) = \mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - g_2(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We prove that u is convex and strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, g_1 and g_2 satisfies

$$\begin{cases} g_1(z) = A_1(az+b) + \overline{A_2}(cz+d) \\ g_2(z) = A_2(az+b) - \overline{A_1}(cz+d) \end{cases}$$

(for each $z \in \mathbb{C}$ with $a, b, d \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$), or

$$\begin{cases} g_1(z) = A_1(a_1 z + b_1) + \overline{A_2} e^{(c_1 z + d_1)} \\ g_2(z) = A_2(a_1 z + b_1) - \overline{A_1} e^{(c_1 z + d_1)} \end{cases}$$

(for each $z \in \mathbb{C}$, where $a_1, b_1, d_1 \in \mathbb{C}$ and $c_1 \in \mathbb{C}\backslash\{0\}$).
However, the number of the absolute values implies that $n = 1$. The great differences between the classes of functions (convex and strictly psh) and strictly convex is one of the purpose of this paper.
Moreover, if we replace $\mathbb{C}^n$ by a convex domain bounded on $\mathbb{C}^n$, the above result is not true.

We show extension results of ([3], Corollaire 17), which is the following.
Let $\alpha, \beta \in \mathbb{C}$, $(\alpha \neq \beta)$ and $g : \mathbb{C}^n \to \mathbb{C}$ be analytic. Using holomorphic differential equations, we prove that $\mid g + \alpha \mid$ and $\mid g + \beta \mid$ are convex functions over $\mathbb{C}^n$ if and only if g is an affine function on $\mathbb{C}^n$.
Observe that the complex structure plays a key role in this situation. For example, let $\varphi(z) = x_1^2 + 1$, for $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $z_1 = (x_1 + iy_1) \in \mathbb{C}$, where $x_1, y_1 \in \mathbb{R}$. Then φ is real analytic on $\mathbb{C}^n$. $\mid \varphi + 0 \mid = \mid \varphi \mid$ and $\mid \varphi + 1 \mid$ are convex functions on $\mathbb{C}^n$. But φ is not affine on $\mathbb{C}^n$.

Let U be a domain of $\mathbb{R}^d, (d \geq 2)$. We denote by sh(U) the subharmonic functions on U and m_d the Lebesgue measure on $\mathbb{R}^d$. Let $f : U \to \mathbb{C}$ be a function. $\mid f \mid$ is the modulus of f, Re(f) is the real part of f. supp(f) is the support of f. For $N \geq 1$ and $h = (h_1, ..., h_N)$, where $h_1, ..., h_N : U \to \mathbb{C}$, $\parallel h \parallel = (\mid h_1 \mid^2 + ... + \mid h_N \mid^2)^{\frac{1}{2}}$.
Let $g : D \to \mathbb{C}$ be an analytic function, D is a domain of $\mathbb{C}$. We denote by $g^{(0)} = g, g^{(1)} = g'$ is the holomorphic derivative of g over D. $g^{(2)} = g'', g^{(3)} = g'''$. In general $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ is the holomorphic derivative of g of order m, for all $m \in \mathbb{N}$.
Let $z \in \mathbb{C}^n, z = (z_1, ..., z_n), n \geq 1$. For $n \geq 2$ and $j \in \{1, ..., n\}$, we write $z = (z_j, Z_j) = (z_1, ..., z_{j-1}, z_j, z_{j+1}, ..., z_n)$ where $Z_j = (z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \mathbb{C}^{n-1}$. If $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, we denote $< z/\xi > = z_1\overline{\xi_1} + ... + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \parallel \zeta - \xi \parallel < r\}$ for $r > 0$, where $\sqrt{< \xi/\xi >} = \parallel \xi \parallel$ is the Euclidean norm of ξ.
$C(U) = \{\varphi : U \to \mathbb{C}/\varphi \text{ is continuous on } U\}$.
$C^k(U) = \{\varphi : U \to \mathbb{C}/\varphi$ is of class C^k on $U\}$ and $C_c^k(U) = \{\varphi : U \to \mathbb{C}/\varphi \in C^k(U)$ and have a compact support on $U\}$, $k \in \mathbb{N} \cup \{\infty\}$ and $k \geq 1$.
Let $\varphi : U \to \mathbb{C}$ be a function of class C^2. $\Delta(\varphi)$ is the Laplacian of φ.
Let D be a domain of $\mathbb{C}^n, (n \geq 1)$. $psh(D)$ and $prh(D)$ are respectively the class of plurisubharmonic and pluriharmonic functions on D.

Definition 1. Let $\varphi : D \to \mathbb{R}$ be a function of class C^2 and $a \in D$. We say that φ is strictly plurisubharmonic at a if $\displaystyle\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(a)\alpha_j \overline{\alpha_k} > 0$, for all $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \setminus \{0\}$.

Moreover, we say that φ is strictly plurisubharmonic on D if φ is strictly psh at every point $a \in D$.
For all $a \in \mathbb{C}, \mid a \mid$ is the modulus of a. Re(a) is the real part of a. $D(a, r) = \{z \in \mathbb{C}/ \mid z - a \mid < r\}$ and $\partial D(a, r) = \{z \in \mathbb{C}/ \mid z - a \mid = r\}$, for $r > 0$.
For p an analytic polynomial over $\mathbb{C}$, deg(p) is the degree of p.

For the study of properties and extension problems of analytic and plurisubharmonic functions we cite the references [1], [4], [5], [6], [7], [8], [10], [13], [14], [15], [16], [19], [20], [21], [24], [25], [26], [27], [29], [30], [32], [34], [35] and [12]. Several properties of analytic functions and their graphs are obtained in [12] and [13].
The class of n-harmonic functions is introduced by Rudin in [33]. There are many investigations of plurisubharmonic functions in [2], [18], [22], [23], [28], [29], [31], [11] and [9]. Good references for the study of convex functions in complex convex domains are [17], [21] and [35].

2. A Fundamental Properties over $\mathbb{C}^n$

The following 4 lemmas (Lemma 1, Lemma 2, Lemma 3 and Lemma 4) are fundamental in this paper. Convex and plurisubharmonic functions are connected by the

Lemma 1. *Let $u : \mathbb{C}^n \to \mathbb{R}$ be a continuous function, $n \geq 1$. Put $v(z,w) = u(w - \overline{z})$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. For $z = (z_1, ..., z_n), \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ and $1 \leq j \leq n$, we write $z_j = (x_j + ix_{j+n})$ and $\alpha_j = (b_j + ib_{j+n})$, where $x_j, x_{j+n}, b_j, b_{j+n} \in \mathbb{R}$.*
The following conditions are equivalent

(a) *u is convex on $\mathbb{C}^n$;*

(b) *v is psh on $\mathbb{C}^n \times \mathbb{C}^n$;*

(c) *For all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$, we have*

$$\frac{1}{2}\sum_{j,k=1}^{2n} \int u(z)\frac{\partial^2 \varphi}{\partial x_j \partial x_k}(z) b_j b_k dm_{2n}(z) = \mathrm{Re}(\sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\alpha_j \alpha_k dm_{2n}(z)) + \sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j \overline{\alpha_k} dm_{2n}(z) \geq 0$$

for each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$;

(d) *For all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$, we have*

$$\mathrm{Re}(\sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\alpha_j \alpha_k dm_{2n}(z)) \leq \frac{1}{4}\sum_{j,k=1}^{2n} \int u(z)\frac{\partial^2 \varphi}{\partial x_j \partial x_k}(z) b_j b_k dm_{2n}(z) \leq \sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j \overline{\alpha_k} dm_{2n}(z)$$

for each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$. (This is an important property in real and complex analysis);

(e) $$\left|\sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\alpha_j \alpha_k dm_{2n}(z)\right| \leq \sum_{j,k=1}^{n} \int u(z)\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j \overline{\alpha_k} dm_{2n}(z),$$
for each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, for each $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$.

Proof. (a) implies (b) is evident.

(b) implies (a).
Case 1. $n = 1$.
Let $\rho : \mathbb{C} \to \mathbb{R}_+$, ρ is a radial C^∞ function, $\mathrm{supp}(\rho) \subset D(0,1)$ and $\int \rho(\xi) dm_2(\xi) = 1$.
For all $\delta > 0$, we define ρ_δ by $\rho_\delta(\xi) = \frac{1}{\delta^2}\rho(\frac{\xi}{\delta})$, for $\xi \in \mathbb{C}$.
Observe that $v(z, .)$ is sh and continuous on $\mathbb{C}$.

Fix $\delta > 0$ and $z \in \mathbb{C}$. We have

$$v(z,.) * \rho_\delta(w) = \int v(z, w-\xi)\rho_\delta(\xi)dm_2(\xi) = \int u(w-\xi-\overline{z})\rho_\delta(\xi)dm_2(\xi)$$
$$= \varphi_\delta(w-\overline{z}) = \psi_\delta(z,w),$$

where $\varphi_\delta(\zeta) = \int u(\zeta-\xi)\rho_\delta(\xi)dm_2(\xi) = u * \rho_\delta(\zeta)$, for $\zeta \in \mathbb{C}$.
Therefore the function φ_δ is C^∞ on $\mathbb{C}$. Consequently, ψ_δ is C^∞ on $\mathbb{C}^2$.

Let $A(z,w,\xi) = v(z,w-\xi)\rho_\delta(\xi)$, for $z,w,\xi \in \mathbb{C}$. Since u is continuous on $\mathbb{C}$, then A is continuous on $\mathbb{C}^3$. Note that the function $A(.,.,\xi)$ is psh on $\mathbb{C}^2$, for each $\xi \in \mathbb{C}$. Since ρ_δ have a support compact, then by ([32], p.75), ψ_δ is psh on $\mathbb{C}^2$.
Consequently, ψ_δ is C^∞ and psh over $\mathbb{C}^2$.
By ([3], Lemme 3 p. 339), the function φ_δ is convex over $\mathbb{C}$. Thus $u * \rho_\delta$ is a convex function on $\mathbb{C}$, for all $\delta > 0$. The sequence of functions $(u * \rho_{\frac{1}{j}})$, (for $j \in \mathbb{N}\backslash\{0\}$), converges to the function u uniformly over all compact subset of $\mathbb{C}$ because u is continuous. Therefore, u is convex on $\mathbb{C}$.

Case 2. $n \geq 2$. This proof is similar to the Case 1.

(a) implies (c) is well known.

(c) implies (a).
Let $j \in \{1,...,2n\}$. If $b_j = 1$ and $b_k = 0$, for all $k \neq j$, then

$$\int u(z)\frac{\partial^2\varphi}{\partial x_j^2}(z)dm_{2n}(z) \geq 0.$$

It follows that

$$\sum_{j=1}^{2n}\int u(z)\frac{\partial^2\varphi}{\partial x_j^2}(z)dm_{2n}(z) = \int u(z)\Delta\varphi(z)dm_{2n}(z) \geq 0,$$

for all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$.
Therefore $u = v$ on $\mathbb{C}^n\backslash E$, where v is a subharmonic function on $\mathbb{C}^n$ and E is a borelien subset of $\mathbb{C}^n$ with $m_{2n}(E) = 0$.

Now, assume that u is not subharmonic on $\mathbb{C}^n$. Then there exists $z_0 \in \mathbb{C}^n$ and $r > 0$ such that

$$u(z_0) > \frac{1}{m_{2n}(B(z_0,r))}\int_{B(z_0,r)} u(\xi)dm_{2n}(\xi).$$

Since

$$\int_{B(z_0,r)} u(\xi)dm_{2n}(\xi) = \int_{B(z_0,r)} v(\xi)dm_{2n}(\xi),$$

it follows that $u(z_0) > v(z_0)$ and consequently, $v(z_0) - u(z_0) < 0$.

Since u is continuous on $\mathbb{C}^n$, then $(v-u)$ is an upper semi-continuous function on $\mathbb{C}^n$. Therefore, there exists $\eta \in]0,r[$ such that $(v-u) < 0$ on $B(z_0,\eta)$. Since $m_{2n}(B(z_0,\eta)) > 0$ and $u = v$ on $\mathbb{C}^n\backslash E$, we have a contradiction.

The rest of the proof of this lemma is similar to the two above proofs. □

Remark 1. The constant $\frac{1}{4}$ is the good constant for the two inequalities in the assertion (d) at Lemma 1.

Let D be a not empty convex domain of $\mathbb{C}^n$, $n \geq 1$ and $s \in \mathbb{N}\backslash\{0,1\}$. There does not exists a constant $c > 0$ such that for all $u : D \to \mathbb{R}$ be a function of class C^s and convex on D, we have

$$\frac{1}{c} \mid \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid \leq \sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_k}(z) b_j b_k \leq c \mid \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid,$$

$\forall z = (z_1, ..., z_n) \in D$, $\forall \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, $z_j = (x_j + ix_{j+n})$, $\alpha_j = (b_j + ib_{j+n})$, $(x_j, x_{j+n}, b_j, b_{j+n} \in \mathbb{R})$, $1 \leq j \leq n$.

Lemma 2. *Let $a, b, c \in \mathbb{C}$. We have*

(A) $(a\alpha\overline{\alpha} + b\beta\overline{\beta} + 2\text{Re}(c\alpha\overline{\beta}) \geq 0$, *for all* $(\alpha, \beta) \in \mathbb{C}^2)$ *if and only if* $(a \geq 0$, $b \geq 0$ *and* $\mid c \mid^2 \leq ab)$.

(B) $(a\alpha\overline{\alpha} + b\beta\overline{\beta} + 2\text{Re}(c\alpha\overline{\beta}) > 0$, *for all* $(\alpha, \beta) \in \mathbb{C}^2\backslash\{0\})$ *if and only if* $(a > 0$, $b > 0$ *and* $\mid c \mid^2 < ab)$.

Proof. See ([3], Lemme 9, p. 354).

Lemma 3. *Let $u : G \to \mathbb{R}$ and $h : D \to \mathbb{C}$, G is a convex domain of $\mathbb{C}^n$, D is a domain of $\mathbb{C}^n$, $n \geq 1$. Suppose that u is a function of class C^2 on G and h is a pluriharmonic (prh) function over D. Then we have*

(A) *The Levi hermitian form of* $\mid h \mid^2$ *is*

$$L(\mid h \mid^2)(z)(\alpha) = \sum_{j,k=1}^{n} \frac{\partial^2(\mid h \mid^2)}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

$$= \mid \sum_{j=1}^{n} \frac{\partial h}{\partial z_j}(z)\alpha_j \mid^2 + \mid \sum_{j=1}^{n} \frac{\partial \overline{(h)}}{\partial z_j}(z)\alpha_j \mid^2$$

for each $z = (z_1, ..., z_n) \in D$, *for all* $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
We can also study the case where h is n-harmonic on D.

(B) *u is convex on G if and only if*

$$\mid \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid \leq \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z \in G$ *and all* $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
u is strictly convex on G if and only if

$$\mid \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid < \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z \in G$ *and every* $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n\backslash\{0\}$.

Proof. Let $z = (z_1, ..., z_n) \in D$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
$\forall j, k \in \{1, ..., n\}$, since h is prh on D then

$$\frac{\partial^2(\mid h \mid^2)}{\partial z_j \partial \overline{z_k}}(z) = \frac{\partial h}{\partial z_j}(z)\frac{\partial(\overline{h})}{\partial \overline{z_k}}(z) + \frac{\partial h}{\partial \overline{z_k}}(z)\frac{\partial(\overline{h})}{\partial z_j}(z).$$

Therefore,

$$\sum_{j,k=1}^{n} \frac{\partial^2(\mid h \mid^2)}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k} = \sum_{j,k=1}^{n} \frac{\partial h}{\partial z_j}(z)\alpha_j\frac{\partial(\overline{h})}{\partial \overline{z_k}}(z)\overline{\alpha_k} + \sum_{j,k=1}^{n} \frac{\partial h}{\partial \overline{z_k}}(z)\overline{\alpha_k}\frac{\partial(\overline{h})}{\partial z_j}(z)\alpha_j$$

$$= (\sum_{j=1}^{n} \frac{\partial h}{\partial z_j}(z)\alpha_j)(\overline{\sum_{k=1}^{n} \frac{\partial(h)}{\partial z_k}(z)\alpha_k}) + (\sum_{j=1}^{n} \frac{\partial \overline{h}}{\partial z_j}(z)\alpha_j)(\overline{\sum_{k=1}^{n} \frac{\partial \overline{h}}{\partial z_k}(z)\alpha_k})$$

$$= \mid \sum_{j=1}^{n} \frac{\partial h}{\partial z_j}(z)\alpha_j \mid^2 + \mid \sum_{j=1}^{n} \frac{\partial \overline{h}}{\partial z_j}(z)\alpha_j \mid^2 .$$

□

The following lemma plays a classical role on several problems of complex analysis. Several fundamental properties of pluripotential theory deduced by this lemma was obtained in this paper.

Lemma 4. *Let* $f_1, ..., f_N, g_1, ..., g_N : D \to \mathbb{C}$*,* D *is a domain of* $\mathbb{C}^n$*,* $n, N \geq 1$*.*
Put $f = (f_1, ..., f_N)$*,* $g = (g_1, ..., g_N)$ *and assume that* $f_1, ..., f_N$*,* $g_1, ..., g_N$ *are holomorphic functions on* D*. Let* $u : D \to \mathbb{R}$ *be a function of class* C^2*. Then* $(\parallel f \parallel^2 + \parallel g \parallel^2)$ *and* $(\parallel f + \overline{g} \parallel^2)$ *have the same hermitian Levi form over* D*. In particular* $(u + \parallel f \parallel^2 + \parallel g \parallel^2)$ *is strictly psh on* D *if and only if* $(u + \parallel f + \overline{g} \parallel^2)$ *is strictly psh on* D*.*

Proof. $\parallel f + \overline{g} \parallel^2 = \sum_{j=1}^{N} \mid f_j + \overline{g_j} \mid^2 = \parallel f \parallel^2 + \parallel g \parallel^2 + \sum_{j=1}^{N} \overline{f_j}\overline{g_j} + \sum_{j=1}^{N} f_j g_j$ on D.

Observe that $\sum_{j=1}^{N}(f_j g_j + \overline{f_j}\overline{g_j}) = 2\mathrm{Re}(\sum_{j=1}^{N} f_j g_j)$ is a pluriharmonic (prh) function on D. Consequently, the Levi hermitian form of the function $\sum_{j=1}^{N}(f_j g_j + \overline{f_j}\overline{g_j})$ is equal zero on $D \times \mathbb{C}^n$. It follows that $\parallel f + \overline{g} \parallel^2$ and $(\parallel f \parallel^2 + \parallel g \parallel^2)$ have the same hermitian Levi form on D. □

Now we choose a proof which is classical in complex analysis of the following.

Theorem 1. *Let* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ *be two analytic functions,* $n \geq 1$ *and* $A_1, A_2 \in \mathbb{C} \backslash \{0\}$*. Put*

$$u_{(a,b)}(z) = \mid A_1(< z/a > + b) - g_1(z) \mid^2 + \mid A_2(< z/a > + b) - g_2(z) \mid^2 = u(z),$$

for $z \in \mathbb{C}^n$, $a \in \mathbb{C}^n$ *and* $b \in \mathbb{C}$.
The following conditions are equivalent

(A) $u_{(a,b)}$ *is convex on* $\mathbb{C}^n$, *for all* $a \in \mathbb{C}^n$ *and* $b \in \mathbb{C}$;

(B)

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2}(< z/c_1 > +d_1)^m \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1}(< z/c_1 > +d_1)^m \end{cases}$$

(*for each* $z \in \mathbb{C}^n$ *with* $a_1, c_1 \in \mathbb{C}^n, b_1, d_1 \in \mathbb{C}, m \in \mathbb{N}$), *or*

$$\begin{cases} g_1(z) = A_1(< z/a_2 > +b_2) + \overline{A_2}e^{(<z/c_2>+d_2)} \\ g_2(z) = A_2(< z/a_2 > +b_2) - \overline{A_1}e^{(<z/c_2>+d_2)} \end{cases}$$

(*for each* $z \in \mathbb{C}^n$, *where* $a_2, c_2 \in \mathbb{C}^n, b_2, d_2 \in \mathbb{C}$).

Proof. Case 1. $n = 1$.
(A) implies (B). For $a, b \in \mathbb{C}$, $u_{(\overline{a},b)}$ is a function of class C^∞ on $\mathbb{C}^2$. Therefore we have

$$\mid \frac{\partial^2 u_{(\overline{a},b)}}{\partial z^2}(z) \mid \leq \frac{\partial^2 u_{(\overline{a},b)}}{\partial z \partial \overline{z}}(z), \quad \forall z \in \mathbb{C}, \forall (a,b) \in \mathbb{C}^2.$$

Fix $z \in \mathbb{C}$. Then

$$\begin{gathered} \mid g_1''(z)[\overline{A_1(az+b) - g_1(z)}] + g_2''(z)[\overline{A_2(az+b) - g_2(z)}] \mid \\ \leq \mid A_1 a - g_1'(z) \mid^2 + \mid A_2 a - g_2'(z) \mid^2, \end{gathered}$$

for all $a, b \in \mathbb{C}$.

<u>State 1</u>. Take $a = 0$. Then

$$\mid -g_1''(z)\overline{g_1}(z) - g_2''(z)\overline{g_2}(z) + \overline{b}(\overline{A_1}g_1''(z) + \overline{A_2}g_2''(z)) \mid \leq \mid g_1'(z) \mid^2 + \mid g_2'(z) \mid^2,$$

for all $b \in \mathbb{C}$.
If $(\overline{A_1}g_1''(z) + \overline{A_2}g_2''(z)) \neq 0$. Then the subset $\mathbb{C}$ is bounded. A contradiction.
It follows that $(\overline{A_1}g_1'' + \overline{A_2}g_2'') = 0$ over $\mathbb{C}$. Consequently, $(\overline{A_1}g_1 + \overline{A_2}g_2)$ is an affine function on $\mathbb{C}$.

<u>State 2</u>. For all $a \in \mathbb{C}$, we have

$$\begin{gathered} \mid g_1''(z)[\overline{A_1 az - g_1(z)}] + g_2''(z)[\overline{A_2 az - g_2(z)}] \mid \\ \leq \mid A_1 a - g_1'(z) \mid^2 + \mid A_2 a - g_2'(z) \mid^2, \quad \forall z \in \mathbb{C}. \end{gathered}$$

It follows that

$$\mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid \leq \mid A_1 a - g_1'(z) \mid^2 + \mid A_2 a - g_2'(z) \mid^2$$

for each $z \in \mathbb{C}$. We have

$$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid a \mid^2 - 2\mathrm{Re}[\overline{a}(\overline{A_1} g_1'(z) + \overline{A_2} g_2'(z))] + \mid g_1'(z) \mid^2 + \mid g_2'(z) \mid^2 - \mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid \geq 0, \forall z \in \mathbb{C}, \quad \forall a \in \mathbb{C}.$$

Now observe that

$$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid a \mid^2 - 2\mathrm{Re}[\overline{a}(\overline{A_1} g_1'(z) + \overline{A_2} g_2'(z))] + \mid g_1'(z) \mid^2 + \mid g_2'(z) \mid^2 - \mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid = \mid a\sqrt{\mid A_1 \mid^2 + \mid A_2 \mid^2}$$
$$- \frac{1}{\sqrt{\mid A_1 \mid^2 + \mid A_2 \mid^2}}(\overline{A_1} g_1'(z) + \overline{A_2} g_2'(z)) \mid^2 + \frac{-1}{\mid A_1 \mid^2 + \mid A_2 \mid^2} \mid \overline{A_1} g_1'(z) + \overline{A_2} g_2'(z) \mid^2$$
$$+ \mid g_1'(z) \mid^2 + \mid g_2'(z) \mid^2 - \mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid \geq 0$$

for each $a \in \mathbb{C}$.
For $a = \frac{1}{|A_1|^2 + |A_2|^2}(\overline{A_1} g_1'(z) + \overline{A_2} g_2'(z))$, we have

$$\frac{\mid A_2 \mid^2}{\mid A_1 \mid^2 + \mid A_2 \mid^2} \mid g_1'(z) \mid^2 + \frac{\mid A_1 \mid^2}{\mid A_1 \mid^2 + \mid A_2 \mid^2} \mid g_2'(z) \mid^2$$
$$- \frac{2}{\mid A_1 \mid^2 + \mid A_2 \mid^2}\mathrm{Re}[\overline{A_1} A_2 g_1'(z)\overline{g_2'}(z)] - \mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid \geq 0.$$

Thus

$$\frac{1}{\mid A_1 \mid^2 + \mid A_2 \mid^2} \mid A_2 g_1'(z) - A_1 g_2'(z) \mid^2 - \mid g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z) \mid \geq 0$$

for each $z \in \mathbb{C}$. Put $A = \frac{A_1}{A_2} \in \mathbb{C}\backslash\{0\}$.
$\overline{A_1} g_1'' + \overline{A_2} g_2'' = 0$ on $\mathbb{C}$ and then $g_2'' = -\overline{A} g_1''$ over $\mathbb{C}$.
Therefore we have

(1)
$$\frac{1}{(\mid A_1 \mid^2 + \mid A_2 \mid^2)} \mid A_2 g_1'(z) - A_1 g_2'(z) \mid^2 \geq \mid g_1''(z)(g_1(z) - A g_2(z)) \mid$$

for each $z \in \mathbb{C}$.
Since $g_1'' = -\frac{1}{\overline{A}} g_2''$, then

(2)

$$\frac{1}{(\mid A_1 \mid^2 + \mid A_2 \mid^2)} \mid A_2 g_1'(z) - A_1 g_2'(z) \mid^2 \geq$$
$$\mid \frac{1}{\overline{A}} g_2''(z)(g_1(z) - A g_2(z)) \mid = \mid \frac{-1}{A} g_2''(z)(g_1(z) - A g_2(z)) \mid$$

for every $z \in \mathbb{C}$.

(1) implies that

$$| g_1''(z)(g_1(z) - Ag_2(z)) |\leq \frac{1}{(| A_1 |^2 + | A_2 |^2)} | A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for each $z \in \mathbb{C}$.
Then

$$| g_1''(z)(g_1(z) - \frac{A_1}{A_2} g_2(z)) |= \frac{1}{| A_2 |^2} | A_2 g_1''(z)(A_2 g_1(z) - A_1 g_2(z)) |$$
$$\leq \frac{1}{(| A_1 |^2 + | A_2 |^2)} | A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for each $z \in \mathbb{C}$.
Then we obtain the inequality

(3)

$$| A_2 g_1''(z)(A_2 g_1(z) - A_1 g_2(z)) |\leq \frac{| A_2 |^2}{(| A_1 |^2 + | A_2 |^2)} | A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for every $z \in \mathbb{C}$.

Now (2) implies the following inequality

(4)

$$| -A_1 g_1''(z)(A_2 g_1'(z) - A_1 g_2'(z)) |\leq \frac{| A_1 |^2}{(| A_1 |^2 + | A_2 |^2)} | A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for every $z \in \mathbb{C}$.

The sum between the inequalities (3) and (4) implies that

$$| A_2 g_1''(z)(A_2 g_1(z) - A_1 g_2(z)) | + | -A_1 g_2''(z)(A_2 g_1(z) - A_1 g_2(z)) |$$
$$\leq \frac{(| A_1 |^2 + | A_2 |^2)}{(| A_1 |^2 + | A_2 |^2)} | A_2 g_1'(z) - A_1 g_2'(z) |^2 =| A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for each $z \in \mathbb{C}$.
By the triangle inequality we have

$$| (A_2 g_1''(z) - A_1 g_2''(z))(A_2 g_1(z) - A_1 g_2(z)) |\leq| A_2 g_1'(z) - A_1 g_2'(z) |^2$$

for each $z \in \mathbb{C}$.

Now put $\varphi(z) = A_2 g_1(z) - A_1 g_2(z)$, for $z \in \mathbb{C}$.
Note that $\varphi : \mathbb{C} \to \mathbb{C}$, φ is holomorphic over $\mathbb{C}$. φ satisfy the holomorphic differential inequality $| \varphi''\varphi |\leq| \varphi' |^2$ on $\mathbb{C}$. Then $\varphi''\varphi = \gamma(\varphi')^2$, where $\gamma \in \mathbb{C}$, $| \gamma |\leq 1$.
By ([3], Corollaire 14, p. 361; Théorème 22, p. 362) exactly $\gamma \in \{1, \frac{t-1}{t} / t \in \mathbb{N}\backslash\{0\}\}$.

Therefore $\varphi(z) = (az+b)^s$ for all $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$ and $s \in \mathbb{N}$, or $\varphi(z) = e^{(cz+d)}$, for all $z \in \mathbb{C}$, where $c, d \in \mathbb{C}$.

Step 1. $\varphi(z) = (az+b)^s$, for all $z \in \mathbb{C}$. Then $A_2 g_1(z) - A_1 g_2(z) = (az+b)^s$.

Now since $\overline{A_1} g_1''(z) + \overline{A_2} g_2''(z) = 0$, then $\overline{A_1} g_1(z) + \overline{A_2} g_2(z) = a_1 z + b_1$, for all $z \in \mathbb{C}$, where $a_1, b_1 \in \mathbb{C}$. We have the system

$$\begin{cases} A_2 g_1(z) - A_1 g_2(z) = (az+b)^s \\ \overline{A_1} g_1(z) + \overline{A_2} g_2(z) = a_1 z + b_1 \end{cases}$$

for each $z \in \mathbb{C}$.

It follows that $(| A_2 |^2 + | A_1 |^2) g_1(z) = \overline{A_2}(az+b)^s + A_1(a_1 z + b_1)$, and then

$$\begin{cases} g_1(z) = A_1(a_2 z + b_2) + \overline{A_2}(a_3 z + b_3)^s \\ g_2(z) = A_2(a_2 z + b_2) - \overline{A_1}(a_3 z + b_3)^s \end{cases}$$

for each $z \in \mathbb{C}$, where $a_2, b_2, a_3, b_3 \in \mathbb{C}$ and $s \in \mathbb{N}$.

Step 2. $\varphi(z) = e^{(cz+d)}$, for all $z \in \mathbb{C}$.

Then we have by the Step 1, the system

$$\begin{cases} A_2 g_1(z) - A_1 g_2(z) = e^{(cz+d)} \\ \overline{A_1} g_1(z) + \overline{A_2} g_2(z) = a_1 z + b_1 \end{cases}$$

for all $z \in \mathbb{C}$, with $(a_1, b_1 \in \mathbb{C})$.

Then

$$\begin{cases} g_1(z) = A_1(c_1 z + d_1) + \overline{A_2} e^{(c_2 z + d_2)} \\ g_2(z) = A_2(c_1 z + d_1) - \overline{A_1} e^{(c_2 z + d_2)} \end{cases}$$

for each $z \in \mathbb{C}$, where $c_1, d_1, c_2, d_2 \in \mathbb{C}$.

(B) implies (A) is evident.

Case 2. $n \geq 2$.

For $z = (z_1, ..., z_n) \in \mathbb{C}^n$, we write $z = (z_1, Z_1)$, $Z_1 \in \mathbb{C}^{n-1}$, $z_1 \in \mathbb{C}$.

We can prove that $(\overline{A_1} g_1 + \overline{A_2} g_2)$ is an affine function on $\mathbb{C}^n$.

$$\overline{A_1} g_1(z) + \overline{A_2} g_2(z) = < z/a_0 > + b_0, \quad a_0 \in \mathbb{C}^n, \quad b_0 \in \mathbb{C}.$$

Consider the functions $g_1(., Z_1)$, $g_2(., Z_1)$ and we use the problem of fibration as follows. By the Case 1, we have

$$\begin{cases} g_1(z) = A_1[\alpha(Z_1) z_1 + \beta(Z_1)] + \overline{A_2} \varphi(z) \\ g_2(z) = A_2[\alpha(Z_1) z_1 + \beta(Z_1)] - \overline{A_1} \varphi(z), \end{cases}$$

where $\alpha, \beta : \mathbb{C}^{n-1} \to \mathbb{C}$ and $\varphi : \mathbb{C}^n \to \mathbb{C}$.

$$A_2 g_1(z) - A_1 g_2(z) = (| A_1 |^2 + | A_2 |^2) \varphi(z).$$

Then φ is analytic on $\mathbb{C}^n$. Consequently,

$$(\overline{A_1} g_1(z) + \overline{A_2} g_2(z)) = (| A_1 |^2 + | A_2 |^2)[\alpha(Z_1) z_1 + \beta(Z_1)] = < z/a_0 > + b_0$$

for each $z \in \mathbb{C}^n$.

Then α and β are analytic functions. α is constant and β is an affine function on $\mathbb{C}^{n-1}$. Then $\alpha(Z_1)z_1 + \beta(Z_1) =< z/\lambda > +\mu$, $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ $(z = (z_1, Z_1) \in \mathbb{C}^n)$. It follows that $| \varphi |^2$ is convex on $\mathbb{C}^n$. By ([3], Théorème 20, p. 358), the proof is complete. □

Theorem 2. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$ and $A_1, A_2 \in \mathbb{C}\backslash\{0\}$. For all $a \in \mathbb{C}^n$ and $b \in \mathbb{C}$, define*

$$u_{(a,b)}(z) =| A_1(< z/a > +b) - g_1(z) |^2 + | A_2(< z/a > +b) - g_2(z) |^2,$$

$$u_{(a,b,c_1,c_2)}(z) =| A_1(< z/a > +b) - g_1(z) + c_1 |^2 + | A_2(< z/a > +b) - g_2(z) + c_2 |^2,$$

for each $z \in \mathbb{C}^n$.

The following assertions are equivalent

(A) $u_{(a,b)}$ *is strictly convex on $\mathbb{C}^n$, for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$;*

(B) $n = 1$ *and g_1, g_2 are affine functions on $\mathbb{C}$ with the condition $(A_1 g_2' \neq A_2 g_1')$;*

(C) *There exists $c_1, c_2 \in \mathbb{C}$ such that $u_{(a,b,c_1,c_2)}$ is strictly convex on $\mathbb{C}^n$, for every $(a, b) \in \mathbb{C}^n \times \mathbb{C}$.*

Proof. (A) implies (B).

Since $u_{(a,b)}$ is strictly convex on $\mathbb{C}^n$, for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$, then by Theorem 1, we have

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2}(< z/c_1 > +d_1)^m \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1}(< z/c_1 > +d_1)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a_1, c_1 \in \mathbb{C}^n, b_1, d_1 \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = A_1(< z/a_2 > +b_2) + \overline{A_2}e^{(<z/c_2>+d_2)} \\ g_2(z) = A_2(< z/a_2 > +b_2) - \overline{A_1}e^{(<z/c_2>+d_2)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a_2, c_2 \in \mathbb{C}^n, b_2, d_2 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2}(< z/c_1 > +d_1)^m \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1}(< z/c_1 > +d_1)^m \end{cases}$$

for each $z \in \mathbb{C}^n$.

$$u_{(a,b)}(z) =| A_1(< z/a > +b- < z/a_1 > -b_1) - \overline{A_2}(< z/c_1 > +d_1)^m |^2$$
$$+ | A_2(< z/a > +b- < z/a_1 > -b_1) + \overline{A_1}(< z/c_1 > +d_1)^m |^2,$$

where $(a, b) \in \mathbb{C}^n \times \mathbb{C}$.
Choose now $a = a_1$ and $b = b_1$. It follows that

$$u(z) =| < z/c_1 > +d_1 |^{2m} (| A_1 |^2 + | A_2 |^2)$$

and u is strictly convex on $\mathbb{C}^n$.
Thus v is strictly convex on $\mathbb{C}^n$, where $v(z) =| < z/c_1 >|^{2m}$, for $z \in \mathbb{C}^n$. But v is strictly convex on $\mathbb{C}^n$ if and only if $m = 1$, $n = 1$ and $c_1 \in \mathbb{C}\backslash\{0\}$.

$$g_1(z) = A_1(a_1 z + b_1) + \overline{A_2}(c_1 z + d_1) = \alpha_1 z + \beta_1,$$
$$g_2(z) = A_2(a_1 z + b_1) - \overline{A_1}(c_1 z + d_1) = \alpha_2 z + \beta_2,$$

for $z \in \mathbb{C}$, with $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$ and ($\alpha_1 \neq 0$ or $\alpha_2 \neq 0$).
In this case $A_1 g_2' = A_1(A_2 a_1 - \overline{A_1} c_1)$, $A_2 g_1' = A_2(A_1 a_1 + \overline{A_2} c_1)$.
$A_1 g_2' \neq A_2 g_1'$, because $- | A_1 |^2 c_1 \neq | A_2 |^2 c_1$.

Case 2.

$$\begin{cases} g_1(z) = A_1(< z/a_2 > +b_2) + \overline{A_2} e^{(<z/c_2>+d_2)} \\ g_2(z) = A_2(< z/a_2 > +b_2) - \overline{A_1} e^{(<z/c_2>+d_2)} \end{cases}$$

for each $z \in \mathbb{C}$. For $(a, b) \in \mathbb{C}^n \times \mathbb{C}$,

$$u_{(a,b)}(z) = | A_1(< z/a > +b- < z/a_2 > -b_2) - \overline{A_2} e^{(<z/c_2>+d_2)} |^2$$
$$+ | A_2(< z/a > +b- < z/a_2 > -b_2) + \overline{A_1} e^{(<z/c_2>+d_2)} |^2 .$$

Choose now $a = a_2$ and $b = b_2$. It follows that

$$u(z) =| e^{(<z/c_2>+d_2)} |^2 (| A_1 |^2 + | A_2 |^2)$$

and u is strictly convex on $\mathbb{C}^n$. Thus φ is strictly convex on $\mathbb{C}^n$, where $\varphi(z) = | e^{<z/c_2>} |^2$, for all $z \in \mathbb{C}^n$. But now observe that φ is not strictly convex at all point of $\mathbb{C}^n$, for all $n \geq 1$. Therefore this case is impossible.

(B) implies (A) is evident.

(B) implies (C). Note that if

$$u_{(a,b,c_1,c_2)}(z) =| A_1(az + b) - g_1(z) + c_1 |^2 + | A_2(az + b) - g_2(z) + c_2 |^2,$$

$a, b, c_1, c_2 \in \mathbb{C}$, we now prove that

$$0 <| A_1 a - g_1' |^2 + | A_2 a - g_2' |^2, \quad \text{for each} \quad a \in \mathbb{C}.$$

If $a = \frac{g_1'}{A_1} \in \mathbb{C}$ (g_1 is an affine function), then $a \neq \frac{g_2'}{A_2}$, because if $a = \frac{g_2'}{A_2}$, then $\frac{g_1'}{A_1} = \frac{g_2'}{A_2}$ and therefore $A_2 g_1' = A_1 g_2'$. A contradiction.
Consequently, $| A_1 a - g_1' |^2 + | A_2 a - g_2' |^2 > 0$, for every $a \in \mathbb{C}$. It follows that $u_{(a,b,c_1,c_2)}$ is strictly convex on $\mathbb{C}$, for all $(a, b, c_1, c_2) \in \mathbb{C}^4$.

(C) implies (B). By the proof of the assertion (A) implies (B), we have

$$\begin{cases} g_1(z) - c_1 = A_1(< z/\alpha_1 > +\beta_1) + \overline{A_2}(< z/\alpha_2 > +\beta_2)^m \\ g_2(z) - c_2 = A_2(< z/\alpha_1 > +\beta_1) - \overline{A_1}(< z/\alpha_2 > +\beta_2)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\alpha_1, \alpha_2 \in \mathbb{C}^n, \beta_1, \beta_2 \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) - c_1 = A_1(< z/\gamma_1 > +\delta_1) + \overline{A_2} e^{(< z/\gamma_2 > +\delta_2)} \\ g_2(z) - c_2 = A_2(< z/\gamma_1 > +\delta_1) - \overline{A_1} e^{(< z/\gamma_2 > +\delta_2)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\gamma_1, \gamma_2 \in \mathbb{C}^n, \delta_1, \delta_2 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) - c_1 = A_1(< z/\alpha_1 > +\beta_1) + \overline{A_2}(< z/\alpha_2 > +\beta_2)^m \\ g_2(z) - c_2 = A_2(< z/\alpha_1 > +\beta_1) - \overline{A_1}(< z/\alpha_2 > +\beta_2)^m \end{cases}$$

for each $z \in \mathbb{C}^n$.

$$\begin{aligned} u_{(a,b,c_1,c_2)}(z) = &\mid A_1(< z/a > +b- < z/\alpha_1 > -\beta_1) + \overline{A_2}(< z/\alpha_2 > +\beta_2)^m \mid^2 \\ &+ \mid A_2(< z/a > +b- < z/\alpha_1 > -\beta_1) + \overline{A_1}(< z/\alpha_2 > +\beta_2)^m \mid^2 \end{aligned}$$

for each $z \in \mathbb{C}^n$.

Take $a = \alpha_1$, $b = \beta_1$, then we have

$$u_{(a,b,c_1,c_2)} = (\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid< z/\alpha_2 > +\beta_2 \mid^{2m} .$$

Therefore $u_{(a,b,c_1,c_2)}$ is strictly convex on $\mathbb{C}^n$ if and only if $m = n = 1$ and $\alpha_2 \neq 0$. Therefore $(g_1 - c_1)$ and $(g_2 - c_2)$ are affine functions and consequently, g_1 and g_2 are affine functions.

$$g_1(z) = \lambda_1 z + \mu_1 = A_1(\alpha_1 z + \beta_1) + \overline{A_2}(\alpha_2 z + \beta_2) + c_1,$$

$$g_2(z) = \lambda_2 z + \mu_2 = A_2(\alpha_1 z + \beta_1) - \overline{A_1}(\alpha_2 z + \beta_2) + c_2,$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$. Then $(A_1 g_2' \neq A_2 g_1')$.

Case 2.

$$\begin{cases} g_1(z) - c_1 = A_1(< z/\gamma_1 > +\delta_1) + \overline{A_2} e^{(< z/\gamma_2 > +\delta_2)} \\ g_2(z) - c_2 = A_2(< z/\gamma_1 > +\delta_1) - \overline{A_1} e^{(< z/\gamma_2 > +\delta_2)} \end{cases}$$

for each $z \in \mathbb{C}^n$.

We prove that this case is impossible. □

Using the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ ($k : \mathbb{C} \to \mathbb{C}, (\gamma, c) \in \mathbb{C}^2$, k is holomorphic on $\mathbb{C}$), we prove

Theorem 3. *Let $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$ and $n \geq 1$. Given two analytic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$. Put $u_{(a,b)}(z) =\mid A_1(< z/a > +b) - g_1(z) \mid^2 + \mid A_2(< z/a > +b) - g_2(z) \mid^2$, for $z \in \mathbb{C}^n$, $(a, b) \in \mathbb{C}^n \times \mathbb{C}$.*
The following conditions are equivalent

(A) $u_{(a,b)}$ *is strictly convex on $\mathbb{C}^n$, for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$;*

(B) $n = 1$, g_1, g_2 *are affine functions on $\mathbb{C}$ and we have the following 3 cases.*
$A_2 = 0$, $A_1 \neq 0$. *Then* $g_2' \neq 0$.
$A_1 = 0$, $A_2 \neq 0$. *Then* $g_1' \neq 0$.
$A_1 \neq 0$ *and* $A_2 \neq 0$. *Then* $A_2 g_1' \neq A_1 g_2'$.

Proof. If $A_1 \neq 0$ and $A_2 \neq 0$, we use the above Theorem 2.
Now suppose that $A_2 = 0$ and $A_1 \neq 0$. For $(a,b) \in \mathbb{C}^n \times \mathbb{C}$, $u_{(a,b)}$ is C^∞ and strictly convex on $\mathbb{C}^n$. Therefore we have

$$\mid \sum_{j,k=1}^{n} \frac{\partial^2 u_{(a,b)}}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid < \sum_{j,k=1}^{n} \frac{\partial^2 u_{(a,b)}}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z = (z_1, ..., z_n) \in \mathbb{C}^n$, for every $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \backslash \{0\}$.

It follows that for $z = (z_1, ..., z_n)$ fixed on $\mathbb{C}^n$, for $a \in \mathbb{C}^n$ fixed and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \backslash \{0\}$ fixed, we have the inequality

$$(S) \qquad \mid \overline{g_1}(z) \sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \; + \; \overline{g_2}(z) \sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k$$
$$- \overline{A_1}(\overline{< z/a > +b}) \sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \mid < \mid A_1 < \alpha/a > - \sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j \mid^2$$
$$+ \mid \sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j}(z)\alpha_j \mid^2$$

for each $b \in \mathbb{C}$.

Observe that the right expression of the above strict inequality (S) is independent of b. Therefore if $\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \neq 0$, then the subset $\mathbb{C}$ is bounded.
A contradiction. It follows that
$\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k = 0$, for every $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
Since g_1 is a holomorphic function over $\mathbb{C}^n$, then g_1 is an affine function on $\mathbb{C}^n$.

Choose $(a_0, b_0) \in \mathbb{C}^n \times \mathbb{C}$ such that $A_1(< z/a_0 > +b_0) = g_1(z)$, for all $z \in \mathbb{C}^n$. Therefore $u_{(a_0,b_0)}(z) =\mid g_2(z) \mid^2$, for each $z \in \mathbb{C}^n$. Consequently, $\mid g_2 \mid^2$ is strictly convex on $\mathbb{C}^n$. Then, $n = 1$. In particular $\mid g_2 \mid^2$ is convex on $\mathbb{C}$. By ([3], Théorème 20, p. 358) we have $g_2(z) = (\lambda z + \delta)^s$, (for all $z \in \mathbb{C}$, where $\lambda, \delta \in \mathbb{C}$, $s \in \mathbb{N}$), or $g_2(z) = e^{(\lambda_1 z + \delta_1)}$, (for all $z \in \mathbb{C}$, with $\lambda_1, \delta_1 \in \mathbb{C}$).

Case 1. $g_2(z) = (\lambda z + \delta)^s$, for all $z \in \mathbb{C}$.
We have $\mid g_2''(z)g_2(z) \mid<\mid g_2'(z) \mid^2$, for each $z \in \mathbb{C}$. Then $\lambda \neq 0$ and $s = 1$.

$$g_2'(z) = \lambda \neq 0, \quad (z \in \mathbb{C}).$$

Case 2. $g_2(z) = e^{(\lambda_1 z + \delta_1)}$, for each $z \in \mathbb{C}$.
$\mid g_2 \mid^2$ is a function of class C^∞ on $\mathbb{C}$. We prove that $\mid g_2 \mid^2$ is not strictly convex at all point of $\mathbb{C}$. Therefore this case is impossible. □

Corollary 1. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions. For $a \in \mathbb{C}^n$, $b, c \in \mathbb{C}$, put*

$$u_{(a,b,c)}(z) =\mid< z/a > +b - g_1(z) + c \mid^2 + \mid< z/a > +b - g_2(z) \mid^2$$

for $z \in \mathbb{C}^n$. The following conditions are equivalent

(A) $u_{(a,b,c)}$ *is convex on* $\mathbb{C}^n$, *for each* $(a,b,c) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$;

(B) g_1 *and* g_2 *are affine functions on* $\mathbb{C}^n$.

Question. Let $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$ and $n \geq 1$. Find exactly all the analytic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex and u is strictly $(n+1)-$ sh on $\mathbb{C}^n \times \mathbb{C}$, where $v(z,w) =\mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - g_2(z) \mid^2$ and $u(z,w) = v(z,w) + v(\overline{z}, w)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$?

The case of the conjugate of holomorphic functions

Theorem 4. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, where $n \geq 1$. Given $(A_1, A_2) \in (\mathbb{C}\backslash\{0\})^2$ and $u(z,w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent*

(A) *u is convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *We have the two following fundamental representations.*

$$\begin{cases} g_1(z) = \overline{A_1}(< z/a > +b) + A_2(< z/c > +d)^m \\ g_2(z) = \overline{A_2}(< z/a > +b) - A_1(< z/c > +d)^m \end{cases}$$

(*for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n, b, d \in \mathbb{C}, m \in \mathbb{N}$*), *or*

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2 e^{(<z/\gamma>+\delta)} \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1 e^{(<z/\gamma>+\delta)} \end{cases}$$

(*for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n$, $\mu, \delta \in \mathbb{C}$*).

Proof. Let $T(z,w) = (z, \overline{w})$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. T is an $\mathbb{R}-$ linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Therefore, $v = uoT$ is convex on $\mathbb{C}^n \times \mathbb{C}$. But

$$v(z,w) =\mid A_1\overline{w} - \overline{g_1}(z) \mid^2 + \mid A_2\overline{w} - \overline{g_2}(z) \mid^2 =\mid \overline{A_1}w - g_1(z) \mid^2 + \mid \overline{A_2}w - g_2(z) \mid^2$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. By the Theorem 1, we conclude the proof. □

Example. Let $g(z) = z^2 + 2$, $z \in \mathbb{C}$. Put $g_1 = g$, $g_2 = -g$.
Then g_1 and g_2 are analytic functions on $\mathbb{C}$. Let $D = D(2i, \frac{1}{4})$. Define $u(z,w) =\mid w - g_1(z) \mid^2 + \mid w - g_2(z) \mid^2$, $v(z,w) =\mid w - \overline{g_1}(z) \mid^2 + \mid w - \overline{g_2}(z) \mid^2$, $(z,w) \in \mathbb{C}^2$. Then $u(z,w) = v(z,w) = 2(\mid w \mid^2 + \mid g(z) \mid^2)$, $(z,w) \in \mathbb{C}^2$. We have u is strictly convex on $D \times \mathbb{C}$. But we can not write g_1 and g_2 on the form of the above theorem.

Now let $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$. Define $u_1(z,w) =\mid A_1 w - k_1(z) \mid^2 + \mid A_2 w - k_2(z) \mid^2$, $v_1(z,w) =\mid A_1 w - \overline{k_3}(z) \mid^2 + \mid A_2 w - \overline{k_4}(z) \mid^2$, for $(z,w) \in D \times \mathbb{C}$, where $k_1 = \overline{A_2}g$, $k_2 = -\overline{A_1}g$, $k_3 = A_2 g$, $k_4 = -A_1 g$. Note that k_1, k_2, k_3 and k_4 are analytic functions on D. We have $u_1(z,w) = v_1(z,w) = (\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid w \mid^2 + \mid g(z) \mid^2)$, for $(z,w) \in D \times \mathbb{C}$. Then u_1 and v_1 are functions strictly convex on $D \times \mathbb{C}$, but k_1, k_2, k_3 and k_4 are not affine functions on D.

It follows that in all bounded not empty convex domain of $\mathbb{C}^n$ ($n \geq 1$), the above theorem is false.

Theorem 5. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, where $n \geq 1$. Let $(A_1, A_2) \in (\mathbb{C} \backslash \{0\})^2$ and define $v(z,w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent*

(A) *v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1, 2\}$ and we have the following cases:*
If $n = 1$, then

$$\begin{cases} g_1(z) = \overline{A_1}(az+b) + A_2(cz+d)^m \\ g_2(z) = \overline{A_2}(az+b) - A_1(cz+d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}, m \in \mathbb{N}$ with $(m = 0, a \neq 0)$, $(m = 1, (a, c) \neq (0, 0))$, $(m \geq 2, a \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\lambda z+\mu) + A_2 e^{(\gamma z+\delta)} \\ g_2(z) = \overline{A_2}(\lambda z+\mu) - A_1 e^{(\gamma z+\delta)} \end{cases}$$

(for each $z \in \mathbb{C}$, where $\lambda, \mu, \gamma, \delta \in \mathbb{C}, (\lambda, \gamma) \neq (0, 0)$).
If $n = 2$, then

$$\begin{cases} g_1(z) = \overline{A_1}(< z/a > +b) + A_2(< z/c > +d) \\ g_2(z) = \overline{A_2}(< z/a > +b) - A_1(< z/c > +d) \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $a, c \in \mathbb{C}^2$, $b, d \in \mathbb{C}$ with the determinant $\det(a, c) \neq 0$), or

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2 e^{(<z/\gamma>+\delta)} \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1 e^{(<z/\gamma>+\delta)} \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $\lambda, \gamma \in \mathbb{C}^2$, $\mu, \delta \in \mathbb{C}$ with the determinant $\det(\lambda, \gamma) \neq 0$).

Proof. Let $T : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$, $T(z,w) = (z, \overline{w})$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
T is an $\mathbb{R}$ linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$. Then $voT = u$ is convex on $\mathbb{C}^n \times \mathbb{C}$. $u(z,w) =\mid \overline{A_1} w - g_1(z) \mid^2 + \mid \overline{A_2} w - g_2(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
It follows that

$$\begin{cases} g_1(z) = \overline{A_1}(< z/a > +b) + A_2(< z/c > +d)^m \\ g_2(z) = \overline{A_2}(< z/a > +b) - A_1(< z/c > +d)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n$, $b, d \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2 e^{(<z/\gamma>+\delta)} \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1 e^{(<z/\gamma>+\delta)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n$, $\mu, \delta \in \mathbb{C}$).
Case 1.

$$\begin{cases} g_1(z) = \overline{A_1}(< z/a > +b) + A_2(< z/c > +d)^m \\ g_2(z) = \overline{A_2}(< z/a > +b) - A_1(< z/c > +d)^m \end{cases}$$

for each $z \in \mathbb{C}^n$. We have

$$
\begin{aligned}
v(z,w) &= \mid A_1(w - \overline{<z/a>} - \overline{b}) - \overline{A_2}(\overline{<z/c>+d})^m \mid^2 \\
&+ \mid A_2(w - \overline{<z/a>} - \overline{b}) + \overline{A_1}(\overline{<z/c>+d})^m \mid^2 \\
&= (\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid w - \overline{<z/a>} - \overline{b} \mid^2 + \mid <z/c>+d \mid^{2m}),
\end{aligned}
$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Let $v_1(z,w) = \mid w - \overline{<z/a>} - \overline{b} \mid^2 + \mid <z/c>+d \mid^{2m}$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. v and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Note that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. By Lemma 4, v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, where

$$v_2(z,w) = \mid w \mid^2 + \mid <z/a>+b \mid^2 + \mid <z/c>+d \mid^{2m}$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ (v_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$).
But the Levi hermitian form of v_2 is

$$L(v_2)(z,w)(\alpha,\beta) = \mid \beta \mid^2 + \mid <\alpha/a> \mid^2 + m^2 \mid <\alpha/c> \mid^2 \mid <z/c>+d \mid^{2m-2},$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and all $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$.

We have $(L(v_2)(z,w)(\alpha,\beta) > 0, \forall (z,w) \in \mathbb{C}^n \times \mathbb{C}, \forall (\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C} \backslash \{0\})$ if and only if $(\varphi_2(z,\alpha) > 0, \forall z \in \mathbb{C}^n, \forall \alpha \in \mathbb{C}^n \backslash \{0\})$, where

$$\varphi_2(\xi,\delta) = \mid <\delta/a> \mid^2 + m^2 \mid <\delta/c> \mid^2 \mid <\xi/c>+d \mid^{2m-2}$$

for $(\xi,\delta) \in \mathbb{C}^n \times \mathbb{C}^n$.

Step 1. $m = 0$.

Then $\mid <\alpha/a> \mid > 0$, for each $\alpha \in \mathbb{C}^n \backslash \{0\}$. Thus $n = 1$ and $a \in \mathbb{C} \backslash \{0\}$.
In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(az+b) + A_2 \\ g_2(z) = \overline{A_2}(az+b) - A_1 \end{cases}$$

for each $z \in \mathbb{C}$.

Step 2. $m = 1$.

Let $\varphi_3(\alpha) = \varphi_2(z,\alpha) = \mid <\alpha/a> \mid^2 + \mid <\alpha/c> \mid^2$, for $(z,\alpha) \in \mathbb{C}^n \times \mathbb{C}$. Now since we have $\varphi_2(z,\alpha) > 0$, for each $z \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}^n \backslash \{0\}$. Then $\varphi_3(\alpha) = \mid <\alpha/a> \mid^2 + \mid <\alpha/c> \mid^2 > 0$, for every $\alpha \in \mathbb{C}^n \backslash \{0\}$.

Put $a = (\overline{a_1}, ..., \overline{a_n})$, $c = (\overline{c_1}, ..., \overline{c_n})$. Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$. We have $\varphi_3(\alpha) = 0$ if and only if $\alpha = 0$. But $\varphi_3(\alpha) = 0$ is equivalent with $<\alpha/a> = 0$ and $<\alpha/c> = 0$. Therefore

$$\begin{cases} \alpha_1 a_1 + ... + \alpha_n a_n = 0 \\ \alpha_1 c_1 + ... + \alpha_n c_n = 0. \end{cases}$$

Then $\alpha_1(a_1,c_1) + ... + \alpha_n(a_n,c_n) = (0,0) \in \mathbb{C}^2$ ($\mathbb{C}^2$ is considered a complex vector space of dimension 2) if and only if $\alpha_1 = ... = \alpha_n = 0$. Then the set of vectors $\{(a_1,c_1), ..., (a_n,c_n)\}$ is a free family of n vectors of $\mathbb{C}^2$. Therefore $n \leq 2$.

State 1. $n = 1$.

$$|< \alpha/a >|^2 + |< \alpha/c >|^2 = | \alpha a |^2 + | \alpha c |^2 > 0,$$

for each $\alpha \in \mathbb{C}\backslash\{0\}$. Then $(a, c) \neq (0, 0)$. Therefore

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2(\overline{c}z + d) \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1(\overline{c}z + d) \end{cases}$$

for each $z \in \mathbb{C}$. We have

$$v_1(z, w) = | w - a\overline{z} - \overline{b} |^2 + | \overline{c}z + d |^2$$

and

$$v_2(z, w) = | w |^2 + | \overline{a}z + b |^2 + | \overline{c}z + d |^2 .$$

v_2 is strictly psh on $\mathbb{C}^2$ because $| a |^2 + | c |^2 > 0$.

State 2. $n = 2$.
In this case $\{(a_1, c_1), (a_2, c_2)\}$ is a basis of the $\mathbb{C}-$ vector space $\mathbb{C}^2$. It follows that $\{(a_1, a_2), (c_1, c_2)\}$ is a basis of $\mathbb{C}^2$ and consequently, $\{(\overline{a_1}, \overline{a_2}), (\overline{c_1}, \overline{c_2})\} = \{a, c\}$ is a basis of $\mathbb{C}^2$. Then the determinant $\det(a, c) \neq 0$.
In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(< z/a > +b) + A_2(< z/c > +d) \\ g_2(z) = \overline{A_2}(< z/a > +b) - A_1(< z/c > +d) \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $a, c \in \mathbb{C}^2$, $b, d \in \mathbb{C}$ with the determinant $\det(a, c) \neq 0$).

Step 3. $m \geq 2$.

$$\varphi_2(z, \alpha) = |< \alpha/a >|^2 + m^2 \, |< \alpha/c >|^2 |< z/c > +d \, |^{2m-2}, \quad z, \alpha \in \mathbb{C}^n.$$

State 1. $c = 0$.
Then $\varphi_2(z, \alpha) = |< \alpha/a >|^2 > 0$, for every $\alpha \in \mathbb{C}^n\backslash\{0\}$.
It follows that $n = 1$. Consequently, $a \neq 0$. In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2 d^m \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1 d^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$, $b, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).

State 2. $c \neq 0$.
There exists $z_0 \in \mathbb{C}^n$ such that $|< z_0/c > +d \, | = 0$.
Since $(2m - 2) \geq 2$, then $|< z_0/c > +d \, |^{2m-2} = 0$. It follows that $\varphi_2(z_0, \alpha) =$ $|< \alpha/a >|^2 > 0$, for each $\alpha \in \mathbb{C}^n\backslash\{0\}$.
Then $n = 1$ and $a \in \mathbb{C}\backslash\{0\}$. In this case

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2(cz + d)^m \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1(cz + d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$, $c \in \mathbb{C}\backslash\{0\}$, $b, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).
Consequently, for $m \geq 2$ and independently of c, we have in all this step 3, $n = 1$ and

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2(cz + d)^m \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1(cz + d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$, $b, c, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).

Case 2.

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2 e^{(<z/\gamma>+\delta)} \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1 e^{(<z/\gamma>+\delta)} \end{cases}$$

for all $z \in \mathbb{C}^n$.

$$v(z, w) = (| A_1 |^2 + | A_2 |^2)(| w - \overline{< z/\lambda >} - \overline{\mu} |^2 + | e^{(<z/\gamma>+\delta)} |^2),$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Let $u_1(z, w) =| w - \overline{< z/\lambda >} - \overline{\mu} |^2 + | e^{(<z/\gamma>+\delta)} |^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
v and u_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. We have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

Now define

$$u_2(z, w) =| w |^2 + |< z/\lambda > +\mu |^2 + | e^{(<z/\gamma>+\delta)} |^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. u_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. By Lemma 4, we have u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

The Levi hermitian form of u_2 is

$$L(u_2)(z, w)(\alpha, \beta) =| \beta |^2 + |< \alpha/\lambda >|^2 + |< \alpha/\gamma >|^2 | e^{(<z/\gamma>+\delta)} |^2,$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, for all $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$. We have

$$(L(u_2)(z, w)(\alpha, \beta) > 0, \;\; \forall (z, w) \in \mathbb{C}^n \times \mathbb{C}, \;\; \forall (\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}\backslash\{(0, 0)\})$$

if and only if

$$(\varphi_1(z, \alpha) =|< \alpha/\lambda >|^2 + |< \alpha/\gamma >|^2 | e^{(<z/\gamma>+\delta)} |^2 > 0, \;\; \forall z \in \mathbb{C}^n, \;\; \forall \alpha \in \mathbb{C}^n\backslash\{0\}).$$

Now observe that $(\varphi_1(z, \alpha) > 0, \forall z \in \mathbb{C}^n, \forall \alpha \in \mathbb{C}^n\backslash\{0\})$ if and only if $(\theta(z, \alpha) = |< \alpha/\lambda >|^2 + |< \alpha/\gamma >|^2 > 0, \forall \alpha \in \mathbb{C}^n\backslash\{0\})$. But θ is independent of $z \in \mathbb{C}^n$. Therefore, u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $\varphi(\alpha) =|< \alpha/\lambda >|^2 + |< \alpha/\gamma >|^2 > 0$, for all $\alpha \in \mathbb{C}^n\backslash\{0\}$).

By the same method of the Case 1, we prove that $n \leq 2$.

Step 1. $n = 1$. Then $(| \lambda |^2 + | \gamma |^2) > 0$.

Step 2. $n = 2$. Then by the same algebraic method developed in the Case 1, we prove that the determinant $\det(\lambda, \gamma) \neq 0$.

The proof is now finished. □

The complete characterization

Theorem 6. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$. Put $v(z,w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent*

(A) *v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1,2\}$ and we have the following three cases.*
If $A_1, A_2 \in \mathbb{C}\backslash\{0\}$, this situation is studied in the above theorem.
If $A_1 \neq 0$, $A_2 = 0$, then g_1 is affine on $\mathbb{C}^n$, $\mid g_2 \mid^2$ is convex on $\mathbb{C}^n$ and $(\mid g_1 \mid^2 + \mid g_2 \mid^2)$ is strictly psh on $\mathbb{C}^n$.
If $A_1 = 0$, $A_2 \neq 0$, then g_2 is affine, $\mid g_1 \mid^2$ is convex on $\mathbb{C}^n$ and $(\mid g_1 \mid^2 + \mid g_2 \mid^2)$ is strictly psh on $\mathbb{C}^n$.

Corollary 2. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2$. Put $v(z,w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent*

(A) *v is convex strictly psh and not strictly convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1,2\}$, $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$ and we have*

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > + \mu) + A_2(< z/\lambda_1 > + \mu_1)^s \\ g_2(z) = \overline{A_2}(< z/\lambda > + \mu) - A_1(< z/\lambda_1 > + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}$, $s \in \mathbb{N}$ with $(s = 0, n = 1, \lambda = 0)$, or $(s = 1, \lambda_1 = 0, n = 1, \lambda \neq 0)$, or $(s \geq 2, n = 1, \lambda \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda_2 > + \mu_2) + A_2 e^{(< z/\lambda_3 > + \mu_3)} \\ g_2(z) = \overline{A_2}(< z/\lambda_2 > + \mu_2) - A_1 e^{(< z/\lambda_3 > + \mu_3)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda_2, \lambda_3 \in \mathbb{C}^n$, $\mu_2, \mu_3 \in \mathbb{C}$, with $(n = 1, \lambda_2 \neq 0)$, or $(n = 1, \lambda_3 \neq 0)$, or $(n = 2$, the determinant $\det(\lambda_2, \lambda_3) \neq 0)$).

Corollary 3. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2$. Put $v(z,w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent*

(A) *v is convex strictly psh and not strictly convex at all point of $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1,2\}$, $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$ and we have*

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > + \mu) + A_2(< z/\lambda_1 > + \mu_1)^s \\ g_2(z) = \overline{A_2}(< z/\lambda > + \mu) - A_1(< z/\lambda_1 > + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}$, $s \in \mathbb{N}$ with $(n = 1, s = 0, \lambda \neq 0)$, or $(n = 1, s \in \mathbb{N}, \lambda_1 = 0, \lambda \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda_2 > + \mu_2) + A_2 e^{(< z/\lambda_3 > + \mu_3)} \\ g_2(z) = \overline{A_2}(< z/\lambda_2 > + \mu_2) - A_1 e^{(< z/\lambda_3 > + \mu_3)} \end{cases}$$

(for each $z \in \mathbb{C}^n$*, where* $\lambda_2, \lambda_3 \in \mathbb{C}^n, \mu_2, \mu_3 \in \mathbb{C}$*, with* $(n = 1, \lambda_3 \neq 0, \lambda_2 = 0)$*, or* $(n = 2$ *and the determinant* $\det(\lambda_2, \lambda_3) \neq 0))$*.*

In fact we have the following.

Theorem 7. *Let* $n \geq 1$ *and consider two holomorphic functions* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$*. Given* $(A_1, A_2) \in (\mathbb{C}\backslash\{0\})^2$*. Let*

$$\begin{aligned} u(z,w) =& \mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2, v(z,w) = u(z,w) + \mid \overline{A_1} w - g_1(z) \mid^2 \\ &+ \mid \overline{A_2} w - g_2(z) \mid^2, v_1(z,w) = \mid \overline{A_1} w - g_1(z) \mid^2 + \mid \overline{A_2} w - g_2(z) \mid^2 \\ &+ \mid \overline{A_1} w - \overline{g_1}(z) \mid^2 + \mid \overline{A_2} w - \overline{g_2}(z) \mid^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$*. The following conditions are equivalent*

(A) u *is convex on* $\mathbb{C}^n \times \mathbb{C}$ *and* v *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$*;*

(B) u *is convex on* $\mathbb{C}^n \times \mathbb{C}$ *and* v_1 *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$*;*

(C) $n \in \{1, 2\}$ *and we have the following two cases.*
(I)

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > + \mu) + A_2(< z/\lambda_1 > + \mu_1)^s \\ g_2(z) = \overline{A_2}(< z/\lambda > + \mu) - A_1(< z/\lambda_1 > + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$*, where* $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}$*,* $s \in \mathbb{N}$*, with* $(n = 1, \lambda \neq 0)$*, or* $(n = 1, \lambda_1 \neq 0, s = 1)$*, or* $(n = 2, s = 1$ *and the determinant* $\det(\lambda, \lambda_1) \neq 0))$*.*
(II)

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda_2 > + \mu_2) + A_2 e^{(< z/\lambda_3 > + \mu_3)} \\ g_2(z) = \overline{A_2}(< z/\lambda_2 > + \mu_2) - A_1 e^{(< z/\lambda_3 > + \mu_3)} \end{cases}$$

(for each $z \in \mathbb{C}^n$*, where* $\lambda_2, \lambda_3 \in \mathbb{C}^n$*,* $\mu_2, \mu_3 \in \mathbb{C}$*, with* $(n = 1, \lambda_2 \neq 0)$*, or* $(n = 1, \lambda_3 \neq 0)$*, or* $(n = 2$ *and the determinant* $\det(\lambda_2, \lambda_3) \neq 0))$*.*

Proof. This proof is similar to the proof of Theorem 4.

Now we can answer to the following question.

Question. Let $n \geq 1$ and $A_1, A_2 \in \mathbb{C}\backslash\{0\}$. Find all the functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ such that $f_1 \in C(\mathbb{C}^n)$ and

$$\begin{cases} \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \varphi_2 \text{ is convex on } \mathbb{C}^n \times \mathbb{C}, \end{cases}$$

or (for example)

$$\begin{cases} \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \varphi_2 \text{ is convex and strictly psh on } \mathbb{C}^n \times \mathbb{C}, \text{ but not strictly convex on all} \\ \text{not empty open ball of } \mathbb{C}^n \times \mathbb{C}, \end{cases}$$

where

$$\varphi_1(z,w) = \log \mid A_1 w - f_1(z) \mid + \log \mid A_2 w - f_2(z) \mid,$$
$$\varphi_2(z,w) = \mid A_1 w - \overline{f_1}(z) \mid^2 + \mid A_2 w - \overline{f_2}(z) \mid^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Using algebraic methods, we can prove the following theorem:

Theorem 8. *Let $n \geq 1$ and $(A_1, A_2) \in \mathbb{C}^2$. Given $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions. Put $u(z,w) = \mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent*

(A) *u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1,2\}$, $(A_1, A_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$ and we have the following three situations.*

  (1) *$A_1 \neq 0$ and $A_2 = 0$. Then $n = 1$, g_1 is affine, $\mid g_2 \mid^2$ is convex and strictly sh on $\mathbb{C}$.*

  (2) *$A_1 = 0$ and $A_2 \neq 0$. Then $n \in \{1,2\}$, $\mid g_1 \mid^2$ is convex on $\mathbb{C}^n$, g_2 is affine on $\mathbb{C}^n$ and $(\mid g_1 \mid^2 + \mid g_2 \mid^2)$ is strictly psh on $\mathbb{C}^n$.*

  (3) *$A_1, A_2 \in \mathbb{C}\backslash\{0\}$. Then $n \in \{1,2\}$, g_1 and g_2 are affine functions on $\mathbb{C}^n$ and $(\mid g_1 \mid^2 + \mid g_2 \mid^2)$ is strictly psh on $\mathbb{C}^n$.*

3. A Classical Complex Analysis Problem

Let $n, N \geq 1$ and $(A_1, B_1, ..., A_N, B_N \in \mathbb{C}\backslash\{0\})$. For $f_1, g_1, ..., f_N, g_N : \mathbb{C}^n \to \mathbb{C}$, define

$$u_1(z,w) = \mid A_1 w - f_1(z) \mid^2 + \mid B_1 w - g_1(z) \mid^2,$$
$$v_1(z,w) = \mid A_1 w - \overline{f_1}(z) \mid^2 + \mid B_1 w - \overline{g_1}(z) \mid^2, \ldots,$$
$$u_N(z,w) = \mid A_N w - f_N(z) \mid^2 + \mid B_N w - g_N(z) \mid^2,$$
$$v_N(z,w) = \mid A_N w - \overline{f_N}(z) \mid^2 + \mid B_N w - \overline{g_N}(z) \mid^2,$$

$u = (u_1 + ... + u_N)$ and $v = (v_1 + ... + v_N)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Define

$$\varphi_1(z,w) = \log \mid A_1 w - f_1(z) \mid + \log \mid B_1 w - g_1(z) \mid, \ldots,$$
$$\varphi_N(z,w) = \log \mid A_N w - f_N(z) \mid + \log \mid B_N w - g_N(z) \mid, \quad (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

Question. Find all the functions $f_1, g_1, ..., f_N, g_N : \mathbb{C}^n \to \mathbb{C}$ such that $f_1, ..., f_N$ are continuous functions on $\mathbb{C}^n$ and

$$\begin{cases} u_1 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ . \\ . \\ . \\ u_N \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_N \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \text{ and} \end{cases}$$

the function u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$?

Question. Find exactly all the functions $f_1, g_1, ..., f_N, g_N : \mathbb{C}^n \to \mathbb{C}$ such that $f_1, ..., f_N$ are continuous functions on $\mathbb{C}^n$, and

$$\begin{cases} v_1 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ . \\ . \\ . \\ v_N \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_N \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \text{ and} \end{cases}$$

v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$?

Theorem 9. *Let $n \geq 1$, $n+1 = 2q$, $q \in \mathbb{N}$. Let $A_1, B_1, ..., A_q, B_q \in \mathbb{C}\backslash\{0\}$ and $f_1, g_1, ..., f_q, g_q : \mathbb{C}^n \to \mathbb{C}$ be $2q$ analytic functions. Define*

$$u_1(z,w) = \mid A_1 w - f_1(z) \mid^2 + \mid B_1 w - g_1(z) \mid^2, \ldots,$$
$$u_q(z,w) = \mid A_q w - f_q(z) \mid^2 + \mid B_q w - g_q(z) \mid^2$$

and $u = (u_1 + ... + u_q)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
The following conditions are equivalent

(A) *$u_1, ..., u_q$ are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *For each $j \in \{1, ..., q\}$, we have*

$$\begin{cases} f_j(z) = A_j(< z/\lambda_j > + \mu_j) + \overline{B_j}\varphi_j(z) \\ g_j(z) = B_j(< z/\lambda_j > + \mu_j) - \overline{A_j}\varphi_j(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, with $\lambda_j \in \mathbb{C}^n$, $\mu_j \in \mathbb{C}$, $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, $\mid \varphi_j \mid^2$ is a convex function on $\mathbb{C}^n$ and

$$(\lambda_1 - \lambda_2, ..., \lambda_1 - \lambda_q, (\overline{\frac{\partial \varphi_1}{\partial z_1}(a)}, ..., \overline{\frac{\partial \varphi_1}{\partial z_n}(a)}), ..., (\overline{\frac{\partial \varphi_q}{\partial z_1}(a)}, ..., \overline{\frac{\partial \varphi_q}{\partial z_n}(a)}))$$

is a basis of $\mathbb{C}^n$, for all $a \in \mathbb{C}^n$.

(*We can also study the problem u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and not strictly convex on all not empty open ball of $\mathbb{C}^n \times \mathbb{C}, ...$*).

Proof. (A) implies (B). Let $j \in \{1, ..., q\}$. By Theorem 1, we have

$$\begin{cases} f_j(z) = A_j(< z/\lambda_j > + \mu_j) + \overline{B_j}\varphi_j(z) \\ g_j(z) = B_j(< z/\lambda_j > + \mu_j) - \overline{A_j}\varphi_j(z) \end{cases}$$

$\varphi_j : \mathbb{C}^n \to \mathbb{C}$, φ_j is analytic and $\mid \varphi_j \mid^2$ is convex on $\mathbb{C}^n$.
In fact $\varphi_j(z) = (< z/\gamma_j > + \delta_j)^{s_j}$, (for all $z \in \mathbb{C}^n$, where $\gamma_j \in \mathbb{C}^n$, $\delta_j \in \mathbb{C}$, $s_j \in \mathbb{N}$), or $\varphi_j(z) = e^{(< z/a_j > + b_j)}$, for all $z \in \mathbb{C}^n$, with $a_j \in \mathbb{C}^n$, $b_j \in \mathbb{C}$.

We consider in this proof the case where

$$\begin{cases} f_j(z) = A_j(< z/\lambda_j > +\mu_j) + \overline{B_j}(< z/\gamma_j > +\delta_j)^{s_j} \\ g_j(z) = B_j(< z/\lambda_j > +\mu_j) - \overline{A_j}(< z/\gamma_j > +\delta_j)^{s_j} \end{cases}$$

for each $z \in \mathbb{C}^n$ and all $j \in \{1, ..., n\}$ (the proof of the other cases are similar of this proof). Therefore,

$$\begin{aligned} u(z, w) = (\mid A_1 \mid^2 + \mid B_1 \mid^2)[\mid w- < z/\lambda_1 > -\mu_1 \mid^2 + \mid< z/\gamma_1 > +\delta_1 \mid^{2s_1}] + \cdots \\ + (\mid A_q \mid^2 + \mid B_q \mid^2)[\mid w- < z/\lambda_q > -\mu_q \mid^2 + \mid< z/\gamma_q > +\delta_q \mid^{2s_q}], \end{aligned}$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Define

$$\begin{aligned} v(z, w) = \mid w- < z/\lambda_1 > -\mu_1 \mid^2 + \mid< z/\gamma_1 > +\delta_1 \mid^{2s_1} + \cdots \\ + \mid w- < z/\lambda_q > -\mu_q \mid^2 + \mid< z/\gamma_q > +\delta_q \mid^{2s_q}, \end{aligned}$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$. u and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.

We have in fact u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Because this situation, we study the function v.

Let $T : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$, $T(z, w) = (z, w+ < z/\lambda_1 >)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. T is a $\mathbb{C}-$ linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Put $v_1 = voT$. Then v_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.
We have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

$$\begin{aligned} v_1(z, w) = \mid w - \mu_1 \mid^2 + \mid< z/\gamma_1 > +\delta_1 \mid^{2s_1} + \mid w- < z/\lambda_2 - \lambda_1 > -\mu_2 \mid^2 \\ + \mid< z/\gamma_2 > +\delta_2 \mid^{2s_2} +...+ \mid w- < z/\lambda_q - \lambda_1 > -\mu_q \mid^2 \\ + \mid< z/\gamma_q > +\delta_q \mid^{2s_q}, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
The Levi hermitian form of v_1 is

$$\begin{aligned} L(v_1)(z, w)(\alpha, \beta) = \mid \beta \mid^2 +s_1^2 \mid< \alpha/\gamma_1 >\mid^2\mid< z/\gamma_1 > +\delta_1 \mid^{2s_1-2} \\ + \mid \beta- < \alpha/\lambda_2 - \lambda_1 >\mid^2 +s_2^2 \mid< \alpha/\gamma_2 >\mid^2\mid< z/\gamma_2 > +\delta_2 \mid^{2s_2-2} +... \\ + \mid \beta- < \alpha/\lambda_q - \lambda_1 >\mid^2 +s_q^2 \mid< \alpha/\gamma_q >\mid^2\mid< z/\gamma_q > +\delta_q \mid^{2s_q-2}, \end{aligned}$$

for $(z, w), (\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$.

Fix now $(z_0, w_0) \in \mathbb{C}^n \times \mathbb{C}$. Let $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$ with $L(v)(z_0, w_0)(\alpha, \beta) = 0$. Then

$$\begin{cases} \beta = 0 \\ s_1^2 \mid < \alpha/\gamma_1 > \mid^2 \mid < z/\gamma_1 > +\delta_1 \mid^{2s_1-2} = 0 \\ < \alpha/\lambda_2 - \lambda_1 > = 0 \\ s_2^2 \mid < \alpha/\gamma_2 > \mid^2 \mid < z/\gamma_2 > +\delta_2 \mid^{2s_2-2} = 0 \\ . \\ . \\ . \\ < \alpha/\lambda_q - \lambda_1 > = 0 \\ s_q^2 \mid < \alpha/\gamma_q > \mid^2 \mid < z/\gamma_q > +\delta_q \mid^{2s_q-2} = 0. \end{cases}$$

Thus

$$\begin{cases} \beta = 0 \\ < \alpha/\lambda_2 - \lambda_1 > = 0 \\ . \\ . \\ . \\ < \alpha/\lambda_q - \lambda_1 > = 0 \\ s_1^2 \mid < \alpha/\gamma_1 > \mid^2 \mid < z/\gamma_1 > +\delta_1 \mid^{2s_1-2} = 0 \\ . \\ . \\ . \\ s_q^2 \mid < \alpha/\gamma_q > \mid^2 \mid < z/\gamma_q > +\delta_q \mid^{2s_q-2} = 0. \end{cases}$$

Therefore this above system is equivalent with $\beta = 0$ and the system

$$\begin{cases} < \alpha/\lambda_2 - \lambda_1 > = 0 \\ . \\ . \\ . \\ < \alpha/\lambda_q - \lambda_1 > = 0 \\ s_1^2 \mid < \alpha/\gamma_1 > \mid^2 \mid < z/\gamma_1 > +\delta_1 \mid^{2s_1-2} = 0 \\ . \\ . \\ . \\ s_q^2 \mid < \alpha/\gamma_q > \mid^2 \mid < z/\gamma_q > +\delta_q \mid^{2s_q-2} = 0. \end{cases}$$

Consequently, v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if (for each $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$ and every $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, the condition $L(v_1)(z, w)(\alpha, \beta) = 0$ implies that $\alpha = 0$

and $\beta = 0$). Then the system

$$\begin{cases} <\alpha/\lambda_2 - \lambda_1> = 0 \\ . \\ . \\ . \\ <\alpha/\lambda_q - \lambda_1> = 0 \\ s_1^2 \mid <\alpha/\gamma_1>\mid^2 \mid <z/\gamma_1> + \delta_1 \mid^{2s_1-2} = 0 \\ . \\ . \\ . \\ s_q^2 \mid <\alpha/\gamma_q>\mid^2 \mid <z/\gamma_q> + \delta_q \mid^{2s_q-2} = 0 \end{cases}$$

implies that $\alpha = 0$.

Using algebraic methods, we have then $(\lambda_2 - \lambda_1, ..., \lambda_q - \lambda_1, \gamma_1, ..., \gamma_q)$ is a basis of $\mathbb{C}^n = \mathbb{C}^{2q-1}$ and $s_1 = ... = s_q = 1$ ($\mathbb{C}^n$ considered a complex vector space of dimension n). $\square$

Theorem 10. *Let $n = 2q$, $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{N}$. Let $f_1, g_1, ..., f_q, g_q : \mathbb{C}^n \to \mathbb{C}$ be $2q$ holomorphic functions and $A_1, B_1, ..., A_q, B_q \in \mathbb{C}\backslash\{0\}$.*
Define

$$u_j(z,w) = \mid A_j w - \overline{f_j}(z) \mid^2 + \mid B_j w - \overline{g_j}(z) \mid^2, \quad u = (u_1 + ... + u_q),$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $j \in \{1, ..., q\}$. The following conditions are equivalent

(A) *$u_1, ..., u_q$ are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *For every $j \in \{1, ..., q\}$,*

$$\begin{cases} f_j(z) = \overline{A_j}(<z/\lambda_j> + \mu_j) + B_j \varphi_j(z) \\ g_j(z) = \overline{B_j}(<z/\lambda_j> + \mu_j) - A_j \varphi_j(z) \end{cases}$$

(for each $z \in \mathbb{C}^n$, with $\lambda_j \in \mathbb{C}^n$, $\mu_j \in \mathbb{C}$, $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and $\mid \varphi_j \mid^2$ is a convex function on $\mathbb{C}^n$) and

$$(\lambda_1, ..., \lambda_q, (\overline{\frac{\partial \varphi_1}{\partial z_1}(a)}, ..., \overline{\frac{\partial \varphi_1}{\partial z_n}(a)}), ..., (\overline{\frac{\partial \varphi_q}{\partial z_1}(a)}, ..., \overline{\frac{\partial \varphi_q}{\partial z_n}(a)}))$$

is a basis of $\mathbb{C}^n$ for all $a \in \mathbb{C}^n$.

(*We can also study the problem u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and not strictly convex on all not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}, ...$).*

4. Real Convexity and Complex Convexity

Question. An original question of complex analysis is now to find exactly the set of all continuous functions $f_1, ..., f_N : D \to \mathbb{C}$ (D is a convex domain of $\mathbb{C}^n, n \geq 1, N \geq 1$) such that φ is psh on $D \times \mathbb{C}$, where $\varphi(z,w) = \log(| w - f_1(z) |^2 + ... + | w - f_N(z) |^2)$, for $(z,w) \in D \times \mathbb{C}$.

Observe that for $N = 1$, this is exactly all the holomorphic functions over D. But for $N \geq 2$, the set of solution contains several classes of functions.

Example. $N = 2$ and $D = \mathbb{C}^n$. Put

$$k_1(z) = (< z/a > +b) + (\overline{< z/c > +d})^s,$$

$$k_2(z) = (< z/a > +b) - (\overline{< z/c > +d})^s,$$

$z \in \mathbb{C}^n$, $a, c \in \mathbb{C}^n \backslash \{0\}$, $b, d \in \mathbb{C}$, $s \in \mathbb{N}\backslash\{0\}$. k_1, k_2, $\overline{k_1}$ and $\overline{k_2}$ are not holomorphic functions over $\mathbb{C}^n$. The function ψ is psh on $\mathbb{C}^n \times \mathbb{C}$, where $\psi(z,w) = \log(| w - k_1(z) |^2 + | w - k_2(z) |^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Theorem 11. *Let $g_1, g_2, k : \mathbb{C}^n \to \mathbb{C}$ be three analytic functions, $n \geq 1$. Let $(A_1, A_2) \in (\mathbb{C}\backslash\{0\})^2$. Put $u(z,w) =| A_1(w - \overline{k}(z)) - g_1(z) |^2 + | A_2(w - \overline{k}(z)) - g_2(z) |^2$, $v(z,w) =| \overline{A_1} w - \overline{g_1}(z) |^2 + | \overline{A_2} w - \overline{g_2}(z) |^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.*
The following conditions are equivalent

(A) *u is convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *k is an affine function and*

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}(< z/c > +d)^m \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}(< z/c > +d)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n, b, d \in \mathbb{C}$), or

$$\begin{cases} g_1(z) = A_1(< z/\lambda > +\mu) + \overline{A_2} e^{(<z/\gamma>+\delta)} \\ g_2(z) = A_2(< z/\lambda > +\mu) - \overline{A_1} e^{(<z/\gamma>+\delta)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n$, $\mu, \delta \in \mathbb{C}$);

(C) *v is convex on $\mathbb{C}^n \times \mathbb{C}$ and k is an affine function on $\mathbb{C}^n$.*

Theorem 12. *Let $A_1, A_2 \in \mathbb{C}\backslash\{0\}$. Consider three holomorphic functions $g_1, g_2, k : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$. Put*

$$\begin{aligned} v(z,w) &= | A_1(w - \overline{k}(z)) - g_1(z) |^2 + | A_2(w - \overline{k}(z)) - g_2(z) |^2, \\ u(z,w) &= | A_1 w - g_1(z) |^2 + | A_2 w - g_2(z) |^2, \\ u_1(z,w) &= | A_1(w - \overline{k}(z)) |^2 + | A_2(w - \overline{k}(z)) |^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

(A) *v is strictly psh and convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n \in \{1, 2\}$, k is an affine function and*

$$\begin{cases} g_1(z) = A_1(< z/\lambda > +\mu) + \overline{A_2}\varphi(z) \\ g_2(z) = A_2(< z/\lambda > +\mu) - \overline{A_1}\varphi(z) \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $\varphi : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi \mid^2$ is convex on $\mathbb{C}^n$ and $(\mid k \mid^2 + \mid \varphi \mid^2)$ is strictly psh on $\mathbb{C}^n$);

(C)

$$\begin{cases} \mid A_2 g_1 - A_1 g_2 \mid^2 \ \text{is convex on } \mathbb{C}^n, \\ (\overline{A_1} g_1 + \overline{A_2} g_2) \ \text{is affine on } \mathbb{C}^n, \\ k \text{ is affine on } \mathbb{C}^n, \text{ and} \\ \text{the function } (\mid k \mid^2 + \frac{1}{(\mid A_1 \mid^2 + \mid A_2 \mid^2)^2} \mid A_2 g_1 - A_1 g_2 \mid^2) \text{ is strictly psh on } \mathbb{C}^n; \end{cases}$$

(D) *u is convex on $\mathbb{C}^n \times \mathbb{C}$, u_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ and the function $(u + u_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.*
(If $n = 1$, we can study the strict plurisubharmonicity of v and u on a neighborhood of $\partial D(0, 1) \times D(0, 1)$).

Remark 2. Let $A_1, A_2 \in \mathbb{C}\backslash\{0\}$ with $(A_1\overline{A_2} \neq \overline{A_1}A_2)$ and $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two analytic functions. Put $u(z, w) =\mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - g_2(z) \mid^2$, $v(z, w) = \mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2$, for $(z, w) \in \mathbb{C}^2$. If u is strictly psh on $\mathbb{C}^2$, then v is strictly psh on $\mathbb{C}^2$ (and the converse is false).

By a simple study of u and v, we prove that this property is not true for the class of convex functions (respectively strictly psh and convex, strictly convex, strictly psh convex and not strictly convex on all not empty Euclidean open ball of $\mathbb{C}^2, ...$). This is one of the great differences between the above classes of functions.

A good comparison between the subject strictly convex and the concept (convex and strictly psh) can be follows by the following two theorems.

Theorem 13. *Fix $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions, $n \in \mathbb{N}\backslash\{0\}$. Let $(A_1, A_2) \in \mathbb{C}^2$. Define*

$$v(z, w) =\mid A_1 w - \overline{g_1}(z) \mid^2 + \mid A_2 w - \overline{g_2}(z) \mid^2, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

The following conditions are equivalent

(A) *v is strictly convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n = 1$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{(0, 0)\}$ and*

$$\begin{cases} g_1(z) = \overline{A_1}(az + b) + A_2(cz + d) \\ g_2(z) = \overline{A_2}(az + b) - A_1(cz + d) \end{cases}$$

(for each $z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}$, $c \neq 0$).

Theorem 14. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions, $n \in \mathbb{N}\backslash\{0\}$. Let $(A_1, A_2) \in \mathbb{C}^2$. Define*

$$u(z,w) = \mid A_1 w - g_1(z) \mid^2 + \mid A_2 w - g_2(z) \mid^2, \quad (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

The following conditions are equivalent

(A) *u is strictly psh and convex on $\mathbb{C}^n \times \mathbb{C}$, but u is not strictly convex in all not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$n = 1$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{(0,0)\}$ and*

$$\begin{cases} g_1(z) = A_1(az+b) + \overline{A_2} e^{(cz+d)} \\ g_2(z) = A_2(az+b) - \overline{A_1} e^{(cz+d)} \end{cases}$$

for each $z \in \mathbb{C}$, with $a, b, d \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$.

Now one can observe that there exists a great differences between the classes (convex and strictly psh) and strictly convex functions in all of the above two theorems.

The representation theorems for another cases

We begin by

Theorem 15. *Let $k(w) = (aw+b)^m$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$. ($\mid k \mid^2$ is convex on $\mathbb{C}$). Let $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$ and consider two holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$. Define*

$$u(z,w) = \mid A_1 k(w) - g_1(z) \mid^2 + \mid A_2 k(w) - g_2(z) \mid^2, \quad (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

We have

(A) *u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if*

$$\begin{cases} g_1(z) = \overline{A_2}\varphi(z) \\ g_2(z) = -\overline{A_1}\varphi(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, where $\varphi : \mathbb{C}^n \to \mathbb{C}$, φ is holomorphic and $\mid \varphi \mid^2$ is convex on $\mathbb{C}^n$;

(B) *u is convex on $\mathbb{C}^n \times \mathbb{C}$ and $u(.,0)$ is strictly psh on $\mathbb{C}^n$ if and only if $n = 1$ and $\mid \varphi \mid^2$ is strictly sh on $\mathbb{C}$.*

(The same case for $k(w) = e^{(a_1 w + b_1)}$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$).

Observe that, in all not empty convex domain G subset of $\mathbb{C}^n$, $(n \geq 2)$, there exists $K : G \to \mathbb{R}$ be a function of class C^2 such that K is strictly psh on G, but K is not convex in all not empty Euclidean open ball subset of G. For example $K_1(z,w) = \mid w - e^{\overline{z}} \mid^2$, $(z,w) \in \mathbb{C}^2$. K_1 is strictly psh on $\mathbb{C}^2$, but K_1 is not convex in all Euclidean open ball of $\mathbb{C}^2$ (consider $K_1(\overline{z}, w)$).

The converse can be studied and investigated by the following.

Theorem 16. *Let* $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$ *and* $n \geq 1$.
Let $\varphi(w) = (aw + b)^m$, *where* $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$ *(for all* $w \in \mathbb{C}$*) and* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ *be two holomorphic functions. Define*

$$u(z, w) =\mid A_1\varphi(w) - g_1(z) \mid^2 + \mid A_2\varphi(w) - g_2(z) \mid^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. *The following conditions are equivalent*

(A) *u is convex and not strictly psh at all point of* $\mathbb{C}^n \times \mathbb{C}$;

(B) *We have the following two cases*

$$\begin{cases} g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -\overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

(*for every* $z \in \mathbb{C}^n$, *where* $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $s \in \mathbb{N}$ *such that* $(s = 0)$, *or* $(n = 1, \lambda = 0)$, *or* $(n \geq 2)$), *or*

$$\begin{cases} g_1(z) = \overline{A_2}e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -\overline{A_1}e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

(*for each* $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, *such that* $(n = 1, \lambda_1 = 0)$, *or* $(n \geq 2)$).
(*The same situation if* $\varphi(w) = e^{(aw+b)}$, *for* $w \in \mathbb{C}$, *where* $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$).

In general observe that if k is an arbitrary holomorphic function on $\mathbb{C}$, there does not exists $(B_1, B_2) \in \mathbb{C}^2\backslash\{0\}$, there does not exists $n \geq 1$ and $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions such that v is convex on $\mathbb{C}^n \times \mathbb{C}$; $v(z, w) =\mid B_1k(w) - f_1(z) \mid^2 + \mid B_2k(w) - f_2(z) \mid^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The example is given by the following theorem which is fundamental in mathematical analysis.

Theorem 17. *Let* $(A_1, A_2) \in (\mathbb{C}\backslash\{0\})^2$ *and* $n \in \mathbb{N}\backslash\{0\}$. *Define* $p_1(w) = w^3$, $p_2(w) = w^4 + w^2$ *and* $p_3(w) = w^3 + w$, *for* $w \in \mathbb{C}$ *and p be an analytic polynomial over* $\mathbb{C}$, $\deg(p) \leq 2$. *Let* $\varphi = (\varphi_1, \varphi_2)$, *where* $\varphi_1, \varphi_2 : \mathbb{C}^n \to \mathbb{C}$ *be two analytic functions. Define*

$$\begin{aligned} u_\varphi(z, w) &= \mid A_1p_1(w) - \varphi_1(z) \mid^2 + \mid A_2p_1(w) - \varphi_2(z) \mid^2, \\ v_\varphi(z, w) &= \mid A_1p_2(w) - \varphi_1(z) \mid^2 + \mid A_2p_2(w) - \varphi_2(z) \mid^2, \\ \psi_\varphi(z, w) &= \mid A_1p_3(w) - \varphi_1(z) \mid^2 + \mid A_2p_3(w) - \varphi_2(z) \mid^2 \quad \textit{and} \\ \rho_\varphi(z, w) &= \mid A_1p(w) - \varphi_1(z) \mid^2 + \mid A_2p(w) - \varphi_2(z) \mid^2, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. *We have the following four assertions:*

(A) *There exists an infinite number of holomorphic functions* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, $g = (g_1, g_2)$ *and* u_g *is convex on* $\mathbb{C}^n \times \mathbb{C}$.

(B) *There does not exists an holomorphic function* $f = (f_1, f_2)$, *where* $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ *such that* v_f *is convex on* $\mathbb{C}^n \times \mathbb{C}$.

(C) *There does not exists an holomorphic function* $k = (k_1, k_2)$, *where* $k_1, k_2 : \mathbb{C}^n \to \mathbb{C}$ *such that* ψ_k *is convex on* $\mathbb{C}^n \times \mathbb{C}$.

(D) *For all polynomial* p *analytic on* $\mathbb{C}$, $\deg(p) \leq 2$, *there exists always an infinite number of holomorphic functions* $\theta_1, \theta_2 : \mathbb{C}^n \to \mathbb{C}$, $\theta = (\theta_1, \theta_2)$ *and* ρ_θ *is convex on* $\mathbb{C}^n \times \mathbb{C}$.

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