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A Generalization of the Hahn-Banach Theorem in Seminormed Quasilinear Spaces

Sümeyye Çakan and Yılmaz Yılmaz

ABSTRACT: The concept of normed quasilinear spaces which is a generalization of normed linear spaces gives us a new opportunity to study with a similar approach to classical functional analysis. In this study, we introduce the notion of seminormed quasilinear space as a generalization of normed quasilinear spaces and give various auxiliary results and examples. We present an analog of Hahn-Banach theorem, in seminormed quasilinear spaces.

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1. Introduction

Normed quasilinear spaces are introduced by Aseev, [2], in an effort to generalize normed linear spaces. A partial order relation was used to define normed quasilinear spaces. Motivated by [2], and using the framework and the tools given in [2], we developed the analysis in these spaces in [5, 6, 7, 8, 9, 12].

In this paper, we introduce the concept of seminormed quasilinear spaces and mention its some basic properties. Also we state and prove a version of Hahn-Banach theorem, one of the fundamental tools for the application of functional analysis, for seminormed quasilinear spaces.

2. Preliminaries and some results on quasilinear spaces and normed quasilinear spaces

In this section, we present some basic definitions and results that appeared in [2] and [12] and which will be using in the sequel. Let us begin with Aseev's main definition.

Definition 2.1. [2] A set X is called quasilinear space (qls, for short), if a partial order relation "$\preceq$", an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

$$x \preceq x, \tag{2.1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x, \tag{2.7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{2.8}$$

$$1 \cdot x = x, \tag{2.9}$$

$$0 \cdot x = \theta, \tag{2.10}$$

$$(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x, \tag{2.11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha \cdot x \preceq \alpha \cdot y \text{ if } x \preceq y. \tag{2.13}$$

Generally, a qls X with the partial order relation "$\preceq$" is denoted by $(X, \preceq)$. Here, we prefer denote the zero vector of X by θ for clarity.

Every linear space is a qls with the partial order relation "$=$".

The most favorite example of qls which is not a linear space is the set of all nonempty, compact and convex subsets of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and multiplication operation by a real number λ defined by

$$\lambda \cdot A = \{\lambda a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$.

Another one is $\Omega(\mathbb{R})$ which is the set of all nonempty compact subsets of real numbers.

In general, $\Omega(E)$ and $\Omega_C(E)$ stand for the space of all nonempty closed bounded and nonempty convex and closed bounded subsets of any normed linear space E, respectively. Both are nonlinear qls with the inclusion relation and a slight modification of addition operation by

$$A+B=\overline{\{a+b : a\in A,\ b\in B\}}$$

and multiplication operation by a $\lambda\in\mathbb{R}$ defined by $\lambda\cdot A=\{\lambda a : a\in A\}$. Where the closure is taken with respect to the standard topology in $\mathbb{R}$.

Lemma 2.1. [2] *In a qls $(X,\preceq)$, the element θ is minimal, i.e., $x=\theta$ if $x\preceq\theta$.*

Let X be a qls and $Y\subseteq X$. Then Y is called a subspace of X if Y is a qls with the same partial order relation and the restriction of the operations on X to Y .

Theorem 2.1. [12] *Y is a subspace of qls X if and only if $\alpha\cdot x+\beta\cdot y\in Y$ for every $x,y\in Y$ and $\alpha,\beta\in\mathbb{R}$.*

An element $x'\in X$ is called inverse of $x\in X$ if $x+x'=\theta$. Further, if an inverse element exists, then it is unique. An element x possessing inverse is called regular, otherwise is called singular. X_r and X_s stand for the sets of all regular and singular elements in X, respectively, [12].

It will be assumed throughout the text that $-x=(-1)\cdot x$.

Suppose that every element x in a qls X has inverse element $x'\in X$. Then the partial order in X is determined by equality, the distributivity condition in (2.11) holds and consequently, X is a linear space, [2]. In a real linear space, "=" is only way to define a partial order such that the conditions (2.1)-(2.13) hold.

On the other hand, an element $x\in X$ is said to be symmetric if $-x=x$, and X_d denotes the set of all symmetric elements.

X_r, X_d and $X_s\cup\{\theta\}$ are subspaces of X and called regular, symmetric and singular subspaces of X, respectively, [12].

Definition 2.2. [2] Let $(X,\preceq)$ be a qls. A real function $\|.\|_X : X\longrightarrow\mathbb{R}$ is called a norm if the following conditions hold:

$$\|x\|_X>0 \text{ if } x\neq\theta, \tag{2.14}$$

$$\|x+y\|_X\leq\|x\|_X+\|y\|_X\,, \tag{2.15}$$

$$\|\alpha\cdot x\|_X=|\alpha|\,\|x\|_X\,, \tag{2.16}$$

$$\text{if } x\preceq y,\text{ then } \|x\|_X\leq\|y\|_X\,, \tag{2.17}$$

$$\text{if for any } \varepsilon>0 \text{ there exists an element } x_\varepsilon\in X \text{ such that} \tag{2.18}$$

$$x\preceq y+x_\varepsilon \text{ and } \|x_\varepsilon\|_X\leq\varepsilon \text{ then } x\preceq y.$$

A qls X ,with a norm defined on it, is called normed quasilinear space (normed qls, for short).

Let $(X, \preceq)$ be a normed qls. Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x,y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r,\; y \preceq x + a_2^r \text{ and } \|a_i^r\| \leq r,\; i = 1,2\right\}.$$

Since $x \preceq y + (x - y)$ and $y \preceq x + (y - x)$ for any elements $x, y \in X$, the quantity $h_X(x,y)$ is well defined. Also, it is not hard to see that the function h_X satisfies all of the metric axioms and we should note that $h_X(x,y)$ may not equal to $\|x-y\|_X$ if X is a nonlinear qls; however $h_X(x,y) \leq \|x-y\|_X$ is always true for any elements $x, y \in X$. Therefore, we use the metric instead of the norm to discuss topological properties in normed quasilinear spaces. For example, $x_n \to x$ if and only if $h_X(x_n, x) \to 0$ for the sequence (x_n) in a normed qls. Although, always $\|x_n - x\|_X \to 0$ implies $x_n \to x$ in normed quasilinear spaces, $x_n \to x$ may not imply $\|x_n - x\|_X \to 0$.

Let E be a real normed linear space. Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces with the norm defined by

$$\|A\|_\Omega = \sup_{a \in A} \|a\|_E. \tag{2.19}$$

In this case, the Hausdorff metric is defined as usual:

$$h_\Omega(A,B) = \inf\{r \geq 0 : A \subseteq B + S(\theta, r),\; B \subseteq A + S(\theta, r)\},$$

where $S(\theta, r)$ is the closed ball of radius r about $\theta \in X$, [2].

Lemma 2.2. [2] *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

Lemma 2.3. [2] *Let X be a normed qls and n be a positive integer.*

- **a)** *Suppose that $x_n \to x_0$, $y_n \to y_0$ and $x_n \preceq y_n$ for any n. Then $x_0 \preceq y_0$.*
- **b)** *Let $x_n \to x_0$ and $z_n \to x_0$. If $x_n \preceq y_n \preceq z_n$ for any n, then $y_n \to x_0$.*
- **c)** *If $x_n + y_n \to x_0$ and $y_n \to \theta$, then $x_n \to x_0$.*

Definition 2.3. [2] Let $(X, \preceq)$ and $(Y, \preccurlyeq)$ be quasilinear spaces. A mapping $T : X \to Y$ is called a quasilinear operator if it satisfies the following three conditions:

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R}, \tag{2.20}$$

$$T(x_1 + x_2) \preccurlyeq T(x_1) + T(x_2), \tag{2.21}$$

$$\text{if } x_1 \preceq x_2, \text{ then } T(x_1) \preccurlyeq T(x_2). \tag{2.22}$$

If X and Y are linear spaces, then the definition of a quasilinear operator coincides with the usual definiton of linear operator. In this case, condition (2.22) is automatically satisfied.

Definition 2.4. [2] Let X and Y be normed linear spaces. Any mapping from X to $\Omega(Y)$ is called a multivalued mapping.

A quasilinear operator $T : X \to \Omega(Y)$ is called a multivalued quasilinear mapping. In this case, conditions (2.20) and (2.21) take the form

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R},$$

$$T(x_1 + x_2) \subset T(x_1) + T(x_2).$$

Also, condition (2.22) is automatically satisfied.

On the other hand, any quasilinear operator from X to $\Omega(\mathbb{R})$ is called a quasilinear functional.

3. Seminormed quasilinear spaces

In this section, we propose a generalization of normed quasilinear spaces. Let us start the following definition.

Definition 3.1. Let $(X, \preceq)$ be a qls. A real function $p : X \to \mathbb{R}$ is called a seminorm if the following conditions hold:

$$p(x) \geq 0 \text{ if } x \neq \theta, \tag{3.1}$$

$$p(x+y) \leq p(x) + p(y), \tag{3.2}$$

$$p(\alpha \cdot x) = |\alpha|\, p(x), \tag{3.3}$$

$$p(x) \leq p(y) \text{ if } x \preceq y. \tag{3.4}$$

A qls X with a seminorm defined on it, is called seminormed quasilinear space (briefly, seminormed qls).

A seminorm p is called total seminorm (or norm) if the condition

$$\text{“if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that}$$
$$x \preceq y + x_\epsilon \text{ and } p(x_\epsilon) \leq \epsilon \text{ then } x \preceq y\text{”} \tag{3.5}$$

holds.

Note that this definition is inspired from the definition of norm presented by Aseev in [2] and every seminormed (normed) qls is a semimetric (metric) qls.

Proposition 3.1. *Let (X, p) be a seminormed qls. Then the equality*

$$h_X(x,y) = \inf\{r \geq 0 : x \preceq y + a_1^r,\ y \preceq x + a_2^r,\ p(a_i^r) \leq r,\ i = 1,2\} \tag{3.6}$$

defines a semimetric on X. If p is total, h_X becomes a metric.

Proof. First of all, we should note that the quantity h_X is well defined since $x \preceq y+(x-y)$ and $y \preceq x+(y-x)$ for any elements $x, y \in X$.

Assume that $x=y$. Then $x \preceq y$ and $y \preceq x$. According to this,

$$x \preceq y+a_1^r \text{ and } y \preceq x+a_2^r$$

for $a_1^r=a_2^r=\theta$. That implies $h_X(x,y)=0$ since $p(a_1^r)=p(a_2^r)=0$.

Clearly h_X is symmetric. Further, remembering that

$$h_X(x,z)=\inf\left\{r \geq 0: x \preceq z+a_1^r,\ z \preceq x+a_2^r \text{ and } p(a_i^r) \leq \frac{r}{2},\ i=1,2\right\}$$

and

$$h_X(z,y)=\inf\left\{r \geq 0: y \preceq z+b_1^r,\ z \preceq y+b_2^r \text{ and } p(b_i^r) \leq \frac{r}{2},\ i=1,2\right\},$$

we write $x \preceq y+a_1^r+b_2^r$ for every elements a_1^r and b_2^r such that $x \preceq z+a_1^r$ and $z \preceq y+b_2^r$.

Similarly, we can say $y \preceq x+a_2^r+b_1^r$ for every elements a_2^r and b_1^r such that $y \preceq z+b_1^r$ and $z \preceq x+a_2^r$. Since

$$p(a_1^r+b_2^r) \leq p(a_1^r)+p(b_2^r) \leq \frac{r}{2}+\frac{r}{2}=r$$

and

$$p(a_2^r+b_1^r) \leq p(a_2^r)+p(b_1^r) \leq \frac{r}{2}+\frac{r}{2}=r,$$

we get $h_X(x,y) \leq h_X(x,z)+h_X(z,y)$. Because

$$\begin{aligned}
h_X(x,y) &= \inf\{r \geq 0: x \preceq y+a_1^r+b_2^r,\ y \preceq x+a_2^r+b_1^r, \\
&\qquad p(a_1^r+b_2^r) \leq r \text{ and } p(a_2^r+b_1^r) \leq r\} \\
&\leq \inf\left\{r \geq 0: x \preceq z+a_1^r,\ z \preceq x+a_2^r,\ p(a_i^r) \leq \frac{r}{2},\ i=1,2\right\} \\
&+ \inf\left\{r \geq 0: y \preceq z+b_1^r,\ z \preceq y+b_2^r,\ p(b_i^r) \leq \frac{r}{2},\ i=1,2\right\} \\
&= h_X(x,z)+h_X(z,y).
\end{aligned}$$

Hence the equality (3.6) defines a semimetric.

Now let us show that h_X becomes a metric whenever that the seminorm p is total:

Let p be total and $h_X(x,y)=0$. Then for any $\epsilon>0$ there exist elements $x_1^\epsilon, x_2^\epsilon \in X$ such that $x \preceq y+x_1^\epsilon$, $y \preceq x+x_2^\epsilon$ and $p(x_i^\epsilon) \leq \epsilon$, $i=1,2$. Hence the totality condition implies that $x \preceq y$ and $y \preceq x$, that is $x=y$. □

The function h_X defined with the equality in (3.6) is called semimetric (metric) derived from the seminorm (total seminorm) p.

Let h_X be semimetric (metric) derived from the seminorm (total seminorm) p. Then the inequality $h_X(x,y) \leq p(x-y)$ holds for every $x, y \in X$.

Proposition 3.2. *Let (X,p) be a seminormed qls and h_X be semimetric (metric) derived from the seminorm (total seminorm) p. Then we have*

i) $h_X(x+y,z+v) \leq h_X(x,z) + h_X(y,v)$,

ii) $h_X(\alpha \cdot x, \alpha \cdot y) = |\alpha| \, h_X(x,y)$,

iii) $p(x) = h_X(x,\theta)$

for each $\alpha \in \mathbb{R}$ and every $x,y,z,v \in X$.

Proof. Let us show that the inequality *i)* holds. Taking into account the definition of h_X and $\inf A + \inf B \geq \inf A + B$, and using (2.12), we write

$$\begin{aligned}
& h_X(x,z) + h_X(y,v) \\
& = \inf\left\{r \geq 0 : x \preceq z + a_1^r,\ z \preceq x + a_2^r,\ p(a_i^r) \leq r/2,\ i = 1,2\right\} \\
& + \inf\left\{r \geq 0 : y \preceq v + b_1^r,\ v \preceq y + b_2^r,\ p(b_i^r) \leq r/2,\ i = 1,2\right\} \\
& \geq \inf\left\{\begin{array}{c} r \geq 0 : x \preceq z + a_1^r,\ y \preceq v + b_1^r,\ z \preceq x + a_2^r,\ v \preceq y + b_2^r, \\ p(a_i^r) \leq r/2,\ p(b_i^r) \leq r/2,\ i = 1,2 \end{array}\right\} \\
& = \inf\left\{\begin{array}{c} r \geq 0 : x + y \preceq z + v + a_1^r + b_1^r,\ z + v \preceq x + y + a_2^r + b_2^r, \\ p(a_i^r + b_i^r) \leq r,\ i = 1,2 \end{array}\right\} \\
& = h_X(x+y, z+v).
\end{aligned}$$

The equalities *ii)* and *iii)* can be also easily obtained. □

Proposition 3.3. *Let (X,p) be a seminormed qls, $x,y \in X$ and h_X be semimetric (metric) derived from the seminorm (total seminorm) p. Then*

$$h_X(x,\theta) \leq h_X(y,\theta) \text{ if } x \preceq y. \tag{3.7}$$

Further, quasilinear space operations are continuous with respect to the topology induced by h_X.

Proof. Primarily, we say that $x \preceq y$ implies $p(x) \leq p(y)$ since p is seminorm. Considering

$$p(x) = h_X(x,\theta) \text{ and } p(y) = h_X(y,\theta),$$

it is obtained $h_X(x,\theta) \leq h_X(y,\theta)$ whenever $x \preceq y$.

Since the topology derived from the semimetric h_X is first countable topology, to say that addition and scalar multiplication operations are continuous, it will be sufficient to show that these operations are sequentially continuous.

For continuity of addition, let (x_n) and (y_n) be two sequences in X such that $x_n \to x$ and $y_n \to y$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_n \preceq x + a_{1,n}^{\epsilon},\ x \preceq x_n + a_{2,n}^{\epsilon} \text{ and } p\left(a_{i,n}^{\epsilon}\right) \leq \frac{\epsilon}{2},\ i = 1,2$$

and

$$y_n \preceq y + b^{\epsilon}_{1,n},\ y \preceq y_n + b^{\epsilon}_{2,n} \text{ and } p\left(b^{\epsilon}_{i,n}\right) \leq \frac{\epsilon}{2},\ i = 1,2$$

whenever $n \geq N$. Taking into account that p is seminorm and using (2.12), we can write

$$\begin{aligned} x_n + y_n &\preceq x + y + a^{\epsilon}_{1,n} + b^{\epsilon}_{1,n}, \\ x + y &\preceq x_n + y_n + a^{\epsilon}_{2,n} + b^{\epsilon}_{2,n} \end{aligned}$$

and

$$\begin{aligned} p\left(a^{\epsilon}_{1,n} + b^{\epsilon}_{1,n}\right) &\leq p\left(a^{\epsilon}_{1,n}\right) + p\left(b^{\epsilon}_{1,n}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ p\left(a^{\epsilon}_{2,n} + b^{\epsilon}_{2,n}\right) &\leq p\left(a^{\epsilon}_{2,n}\right) + p\left(b^{\epsilon}_{2,n}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

These imply that $x_n + y_n \to x + y$.

Hence, it remains to show that multiplication operation is continuous. Let (x_n) be a sequence in X such that $x_n \to x$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_n \preceq x + a^{\epsilon}_{1,n},\ x \preceq x_n + a^{\epsilon}_{2,n} \text{ and } p\left(a^{\epsilon}_{i,n}\right) \leq \frac{\epsilon}{|\lambda|},\ \lambda \in \mathbb{R}^+,\ i = 1,2$$

whenever $n \geq N$. Using (2.8), (2.12), (2.13) and the fact that p is seminorm, we can say

$$\begin{aligned} \lambda \cdot x_n &\preceq \lambda \cdot x + \lambda \cdot a^{\epsilon}_{1,n}, \\ \lambda \cdot x &\preceq \lambda \cdot x_n + \lambda \cdot a^{\epsilon}_{2,n} \end{aligned}$$

and

$$p\left(\lambda \cdot a^{\epsilon}_{i,n}\right) \leq |\lambda|\, p\left(a^{\epsilon}_{i,n}\right) \leq \epsilon,\ i = 1,2.$$

This implies that $\lambda \cdot x_n \to \lambda \cdot x$. □

Also, we note that the semimetric (metric) h_X induced by a seminorm (total seminorm) on the qls X is not translation invariant. But this semimetric (metric) satisfies the inequality

$$h_X(x + a, y + a) \leq h_X(x, y),\ a \in X.$$

Indeed,

$$h_X(x + a,\ y + a) \leq h_X(x, y) + h_X(a, a) = h_X(x, y).$$

Now let us present an example of seminorm function which is not a norm.

Example 3.1. Consider the qls $\Omega_C(\mathbb{R}^2)$ and the function

$$p(A) = \sup\{|x_2| : (x_1, x_2) \in A\}$$

for any $A \in \Omega_C(\mathbb{R}^2)$.

It is easy to see that p holds seminorm axioms. On the other hand, p is not a norm since $p(A) = 0$, for element $A = \{(t, 0) : -1 \leq t \leq 1\} \in \Omega_C(\mathbb{R}^2) \neq \theta$. Also the condition (2.18) is also not satisfied:

Let $A = \{(t, 0) : 0 \leq t \leq 2\}$, $B = \{(t, 0) : 0 \leq t \leq 1\}$ and $\epsilon > 0$ be arbitrary. Let us define as

$$A_\epsilon = \{(t + \epsilon, 0) : 0 \leq t \leq 1\}.$$

Then $p(A_\epsilon) = 0$ and $A \subset B + A_\epsilon$, but $A \subsetneq B$.

Example 3.2. The function $q(A) = \frac{p(A)}{1+p(A)}$ formed by aid of the seminorm p in Example 3.1 is not a seminorm on $\Omega_C(\mathbb{R}^2)$, since

$$q(\lambda \cdot A) = \frac{p(\lambda \cdot A)}{1 + p(\lambda \cdot A)} = \frac{|\lambda|\, p(A)}{1 + |\lambda|\, p(A)} \neq \lambda q(A).$$

In the following, we give an example of a semimetric map that is not a metric.

Example 3.3. Let $A, B \in \Omega_C(\mathbb{R}^2)$ and

$$d(A, B) = \sup\left\{ \sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B \right\}. \tag{3.8}$$

Firstly let us show that this formula defines a function from $\Omega_C(\mathbb{R}^2)$ to $\mathbb{R}$:

Consider the projection

$$p_1 : \mathbb{R}^2 \to \mathbb{R},\ p_1(a_1, a_2) = a_1$$

and remember that p_1 is continuous. $p_1(A)$ and $p_1(B)$ are compact subsets of $\mathbb{R}$ since A and B are compact in $\mathbb{R}^2$. Hence there exist the numbers $M_1, M_2 \geq 0$ such that $|x| \leq M_1$ for every $x \in p_1(A)$ and $|x| \leq M_2$ for every $x \in p_1(B)$. Therefore, since

$$\begin{aligned} &\sup\left\{ \sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B \right\} \\ &= \sup\left\{ \sqrt{|a_1 - b_1|} : a_1 \in p_1(A),\ b_1 \in p_1(B) \right\} \end{aligned}$$

and $|x| \leq \sqrt{M_1 + M_2}$ for $x \in \left\{ \sqrt{|a_1 - b_1|} : a_1 \in p_1(A),\ b_1 \in p_1(B) \right\}$, the function d is well defined.

It is easy to verify that d is a semimetric. But d is not a metric on $\Omega_C(\mathbb{R}^2)$. Indeed, for elements $A = \{(2, 3)\}$ and $B = \{(2, 4)\}$ in $\Omega_C(\mathbb{R}^2)$, $d(A, B) = 0$, but $A \neq B$.

On the other hand, we can show that the semimetric d defined with (3.8) holds the condition (3.7) and the algebraic operations on $\Omega_C(\mathbb{R}^2)$ are continuous according to this semimetric.

For continuity of addition, let (A_n) and (B_n) be sequences in $\Omega_C(\mathbb{R}^2)$ such that $A_n \to A$, $B_n \to B$ and we take any $x_n \in A_n + B_n$. Then there exist $a_n \in A_n$ and $b_n \in B_n$ such that $x_n = a_n + b_n$.We can write as $a_n = (a_{1,n}, a_{2,n})$ and $b_n = (b_{1,n}, b_{2,n})$ since $A_n, B_n \in \Omega_C(\mathbb{R}^2)$. Because of $A_n \to A$, $B_n \to B$, we have

$$d(A_n, A) = \sup\left\{ \sqrt{|a_{1,n} - a_1|} : (a_{1,n}, a_{2,n}) \in A_n,\ (a_1, a_2) \in A \right\} \to 0$$

and

$$d(B_n, B) = \sup\left\{\sqrt{|b_{1,n} - b_1|} : (b_{1,n}, b_{2,n}) \in B_n,\ (b_1, b_2) \in B\right\} \to 0,$$

whenever $n \to \infty$. Hence, we obtain that

$$\begin{aligned} & d(A_n + B_n, A + B) \\ & = \sup\left\{\sqrt{|a_{1,n} + b_{1,n} - (a_1 + b_1)|} : (a_{1,n} + b_{1,n}, a_{2,n} + b_{2,n}) \in A_n + B_n,\right. \\ & \qquad (a_1 + b_1, a_2 + b_2) \in A + B\} \\ & \leq \sup\left\{\sqrt{|a_{1,n} - a_1|} : (a_{1,n}, a_{2,n}) \in A_n,\ (a_1, a_2) \in A\right\} \\ & + \sup\left\{\sqrt{|b_{1,n} - b_1|} : (b_{1,n}, b_{2,n}) \in B_n,\ (b_1, b_2) \in B\right\} \\ & \to 0 + 0 = 0\ (n \to \infty). \end{aligned}$$

This shows that the addition operation is continuous. Similarly it can be seen that the real-scalar multiplication operation is continuous.

Lastly it remains to show that the condition (3.7) is satisfied. Assume that $A \subseteq B$. Then

$$\begin{aligned} d(A, \theta) &= \sup\left\{\sqrt{|a_1|}\ :\ (a_1, a_2) \in A\right\} \\ &\leq \sup\left\{\sqrt{|a_1|}\ :\ (a_1, a_2) \in B\right\} \\ &= d(B, \theta). \end{aligned}$$

Remark 3.1. Every semimetric on a qls may not be obtained from a seminorm. In Example 3.3, if the semimetric defined on $\Omega_C(\mathbb{R}^2)$ is obtained from a seminorm, the property *ii)* in Proposition 3.2 should hold. However, we see that

$$\begin{aligned} d\left(\lambda \cdot A, \lambda \cdot B\right) &= \sup\left\{\sqrt{|\lambda \cdot a_1 - \lambda \cdot b_1|} : (\lambda \cdot a_1, \lambda \cdot a_2) \in \lambda \cdot A,\ (\lambda \cdot b_1, \lambda \cdot b_2) \in \lambda \cdot B\right\} \\ &= \sqrt{|\lambda|} \sup\left\{\sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B\right\} \\ &= \sqrt{|\lambda|} d\left(A, B\right). \end{aligned}$$

The following proposition is a comment of the condition (2.18).

Proposition 3.4. *Let X be a normed qls, $\mathcal{N}_\theta$ is the family of all neighbourhoods of θ and $x, y \in X$. If for any $V \in \mathcal{N}_\theta$ there exists some $b \in V$ such that $x \preceq y + b$, then $x \preceq y$.*

Remark 3.2. In Proposition 3.4, the hypothesis “Let X be a normed qls” is indispensable. Indeed, Let us recall from Example 3.1 that the function

$$p(A) = \sup\left\{|x_2| : (x_1, x_2) \in A\right\},\ A \in \Omega_C(\mathbb{R}^2)$$

is seminorm on $\Omega_C(\mathbb{R}^2)$. We can construct a topology τ on $\Omega_C(\mathbb{R}^2)$ by aid of p in such a way that

$$U \in \tau \Leftrightarrow \{A : p(A) < \epsilon\} \subseteq U, \text{ for some } \epsilon > 0.$$

We note that τ is a semimetrizable topology with the semimetric

$$d(A, B) = \inf\{r \geq 0 : A \subseteq B + C_1^r,\ B \subseteq A + C_2^r,\ p(C_i^r) \leq r,\ i = 1, 2\}.$$

Now, let $A = \{(t, 0) \in \mathbb{R}^2 : 0 \leq t \leq 2\}$, $B = \{(t, 0) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$, $\epsilon > 0$ be arbitrary and

$$B_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 < \epsilon\}.$$

Then there exists $B_\epsilon \in V$ for every $V \in \mathcal{N}_\theta$ such that $A \subset B + B_\epsilon$. However $A \nsubseteq B$.

Remark 3.3. In Lemma 2.3, the hypothesis "Let X be a normed qls" can not be relaxed. Indeed, let us recall that every linear space is a qls with the partial order relation "=" and consider the element $x = (x_1, x_2)$ and the seminorm

$$p(x) = p\left((x_1, x_2)\right) = \left\{|x_1| : (x_1, x_2) \in \mathbb{R}^2\right\}$$

on the qls $(\mathbb{R}^2, =)$.

Let $(x_n) = \left(\left(\frac{1}{n}, 0\right)\right)_{n=1}^{\infty}$ and $(y_n) = \left(\left(\frac{1}{n}, 0\right)\right)_{n=1}^{\infty}$.

We see that $x_n = y_n$ for every n. On the other hand, the sequence $\left(\left(\frac{1}{n}, 0\right)\right)_{n=1}^{\infty}$ converges to different two elements of $\mathbb{R}^2$ according to this seminorm. For example,

$$(x_n) = \left(\left(\frac{1}{n}, 0\right)\right)_{n=1}^{\infty} \to (0, 1) = x$$

and

$$(y_n) = \left(\left(\frac{1}{n}, 0\right)\right)_{n=1}^{\infty} \to (0, 2) = y$$

since

$$p\left(\left(\frac{1}{n}, 0\right) - (0, 1)\right) = p\left(\left(\frac{1}{n}, -1\right)\right) = \left|\frac{1}{n}\right| \to 0$$

and

$$p\left(\left(\frac{1}{n}, 0\right) - (0, 2)\right) = p\left(\left(\frac{1}{n}, -2\right)\right) = \left|\frac{1}{n}\right| \to 0$$

while $n \to \infty$. However $x \neq y$.

Now, let us denote by $\Omega_C^n(\mathbb{R})$ the family of all $n-$tuples intervals which constitute an important part of interval analysis.

$$\Omega_C^n(\mathbb{R}) = \{X = (X_1, X_2, ..., X_n) : X_i \in \Omega_C(\mathbb{R}) \text{ for } 1 \leq i \leq n\}.$$

We emphasize that $\Omega_C^n(\mathbb{R})$ is different from $\Omega_C(\mathbb{R}^n)$ which is the family of all nonempty closed, bounded and convex subsets of $\mathbb{R}^n$.

$\Omega_C^n(\mathbb{R})$ is a qls with the operations "$\oplus$", "$\odot$" and partial order relation "$\preceq$" defined by

$$\oplus : \Omega_C^n(\mathbb{R}) \times \Omega_C^n(\mathbb{R}) \to \Omega_C^n(\mathbb{R}),$$
$$X \oplus Y = (X_1 + Y_1, X_2 + Y_2, ..., X_n + Y_n)$$

and

$$\odot : \mathbb{R} \times \Omega_C^n(\mathbb{R}) \to \Omega_C^n(\mathbb{R}),$$
$$\alpha \odot X = (\alpha \cdot X_1, \alpha \cdot X_2, ..., \alpha \cdot X_n)$$

and

$$X \preceq Y \Leftrightarrow X_i \subseteq Y_i \text{ for every } i \in \{1, 2, ..., n\}$$

for $X = (X_1, X_2, ..., X_n)$, $Y = (Y_1, Y_2, ..., Y_n) \in \Omega_C^n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.

$\Omega_C^n(\mathbb{R})$ is a seminormed qls with equality defined by

$$\|X\|_{\Omega_C^n(\mathbb{R})} = \|X_i\|_{\Omega_C(\mathbb{R})}$$

for fixed $i \in \{1, 2, ..., n\}$.

For example, on seminormed qls $\Omega_C^3(\mathbb{R})$, the equality

$$\|X\|_{\Omega_C^3(\mathbb{R})} = \|X_1\|_{\Omega_C(\mathbb{R})}$$

defines a seminorm. It is not hard to see that seminorm axioms are hold. The function defined by this way is not a norm since $\|X\| = 0$ for element

$$X = \{[0, 0], [1, 3], [-3, -2]\} \in \Omega_C^3(\mathbb{R}) \neq \theta.$$

Example 3.4. Also the condition (2.18) is also not satisfied:

Let $X = \{[1, 2], [3, 5], [-4, -3]\}$, $Y = \{[1, 2], [4, 6], [3, 5]\}$ and $\epsilon > 0$ be arbitrary. Let us define as

$$X_\epsilon = \{[0, 0], [-2, 1], [-8, -7]\}.$$

Then $\|X_\epsilon\| = 0$ and $X \preceq Y + X_\epsilon$, but $X \npreceq Y$.

We note that T will be called as a linear operator between quasilinear spaces, if T satisfies the following conditions:

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R}, \tag{3.9}$$

$$T(x_1 + x_2) = T(x_1) + T(x_2). \tag{3.10}$$

Also, any linear operator from the quasilinear space X to $\mathbb{R}$ is called a linear functional on quasilinear space X.

The Hahn-Banach theorem is an important tool in functional analysis and there are several versions of it. Let us note that we are largely inspired by Theorem 2.2 in [4], in stating the Hahn-Banach theorem for seminormed quasilinear spaces. The impact of the Hahn-Banach theorem is the existence of linear functionals having specified properties on a quasilinear space. The following is the main result of our work.

Theorem 3.1. *Let p be a seminorm on the quasilinear space X and Y be a subspace of X. Suppose that f is a linear functional from Y to $\mathbb{R}$ and $f(y) \leq p(y)$ for all $y \in Y$. Suppose also that φ is a quasilinear functional from X to $\Omega_C(\mathbb{R})$ and $f(x) \in \varphi(x)$ for every $x \in Y$. Then there exists a linear functional g from X to $\mathbb{R}$ such that $g(x) = f(x)$ for any $x \in Y$ and $g(x) \in \varphi(x)$ for any $x \in X$.*

Proof. Let Z be a subspace of X containing Y, g be a linear functional on Z that extends f, and $g(z) \leq p(z)$ for all $z \in Z$. Also, let $\mathcal{Z}$ be the set of all pairs (Z, g). First of all, since the pair (Y, f) is obviously an element of $\mathcal{Z}$, the set $\mathcal{Z}$ is not empty.

Define a partial order relation "$\ll$" on $\mathcal{Z}$ as follows:

$$(Z_1, g_1) \ll (Z_2, g_2) \Leftrightarrow \begin{cases} Z_1 \subset Z_2 \\ g_2(z) = g_1(z), \text{ for all } z \in Z_1 \end{cases}.$$

Using Zorn Lemma, $\mathcal{Z}$ posesses a maximal totally ordered subset $\{(Z_\alpha, g_\alpha)\}$. If it is defined as $Z = \bigcup Z_\alpha$, clearly, Z is a subspace of X. Also, if $z \in Z$, then $z \in Z_\alpha$ for some α.

If $z \in Z_\alpha$ and $z \in Z_\beta$, then, without loss of generality, we may assume that $(Z_\alpha, g_\alpha) \ll (Z_\beta, g_\beta)$. Therefore $g_\alpha(z) = g_\beta(z)$, so that we may uniquely define $g(z) = g_\alpha(z)$ whenever $z \in Z_\alpha$.

Now, let us show that the function g defined by this way is a linear functional on Z. To do this, let z_1 and z_2 be elements of Z. Then $z_1 \in Z_\alpha$ and $z_2 \in Z_\beta$ for some α and β. Since the set $\{(Z_\gamma, g_\gamma)\}$ is totally ordered, we may assume, again without loss of generality, that $Z_\alpha \subset Z_\beta$, hence both z_1 and z_2 are in Z_β. So

$$g(\lambda_1 \cdot z_1 + \lambda_2 \cdot z_2) = g_\beta(\lambda_1 \cdot z_1 + \lambda_2 \cdot z_2) = \lambda_1 g_\beta(z_1) + \lambda_2 g_\beta(z_2) = \lambda_1 g(z_1) + \lambda_2 g(z_2).$$

We note that if $y \in Y$, then $g(y) = f(y)$, so that g is an extension of f. So, g is a linear functional on the subspace Z, that extends f, for which $g(z) \leq p(z)$ and $g(z) \in \varphi(z)$ for all $z \in Z$, so that the proof will be complete if we show that $Z = X$.

Assume that $Z \neq X$, and v be an element in X which is not in Z. Also, Z' denotes the set of all elements in the form $z + \lambda \cdot v$ for $\lambda \in \mathbb{R}$ and $z \in Z$.

On the other hand, since

$$\theta = (1 - 1) \cdot v \preceq v - v$$

and

$$z + z' \preceq z + z' \text{ for any } z, z' \in Z,$$

we write $z + z' \preceq z + z' + v - v$ from (2.12). Also $p(z + z') \leq p(z + z' + v - v)$ by the fact that p is a seminorm. Therefore, we observe

$$\begin{aligned} g(z) + g(z') &= g(z + z') \\ &\leq p(z + z') \\ &\leq p(z + z' + v - v) \\ &\leq p(z + v) + p(z' - v) \end{aligned}$$

or

$$g(z') - p(z' - v) \leq p(z + v) - g(z)$$

for any $z, z' \in Z$.

Consider the sets

$$W_1 = \{g(z') - p(z' - v) : z' \in Z\} \subset \mathbb{R},$$

$$W_2 = \{p(z + v) - g(z) : z \in Z\} \subset \mathbb{R}$$

and say

$$\sup W_1 = w_1 \text{ and } \inf W_2 = w_2.$$

It is clear that $w_1 \leq w_2$. Take w_0 to be any number for which $w_1 \leq w_0 \leq w_2$ and define g' on Z' by

$$g'(z + \lambda \cdot v) = g(z) + \lambda \cdot w_0.$$

It is easy to see that g' is linear and extends f.

If $\lambda > 0$, then

$$\begin{aligned} g'(z + \lambda \cdot v) &= \lambda \left(g\left(\frac{z}{\lambda}\right) + w_0 \right) \\ &\leq \lambda \left(g\left(\frac{z}{\lambda}\right) + w_2 \right) \\ &\leq \lambda \left(g\left(\frac{z}{\lambda}\right) + p\left(\frac{z}{\lambda} + v\right) - g\left(\frac{z}{\lambda}\right) \right) \\ &= \lambda p\left(\frac{z}{\lambda} + v\right) \\ &= p(z + \lambda \cdot v). \end{aligned}$$

On the other hand, if $\lambda < 0$, then

$$\begin{aligned} g'(z + \lambda \cdot v) &= |\lambda| \left(g\left(\frac{z}{|\lambda|}\right) - w_0 \right) \\ &\leq |\lambda| \left(g\left(\frac{z}{|\lambda|}\right) - w_1 \right) \\ &\leq |\lambda| \left(g\left(\frac{z}{|\lambda|}\right) - g\left(\frac{z}{|\lambda|}\right) + p\left(\frac{z}{|\lambda|} - v\right) \right) \\ &= |\lambda| \, p\left(\frac{z}{|\lambda|} - v\right) \\ &= p(z + \lambda \cdot v). \end{aligned}$$

This proves $g'(z + \lambda \cdot v) \leq p(z + \lambda \cdot v)$ for all $z + \lambda \cdot v \in Z'$. Hence $(Z', g') \in \mathcal{Z}$ and $(Z, g) \ll (Z', g')$. But then the element $(Z', g') \in \mathcal{Z}$ will contradicts with the maximality of (Z, g) by the fact that $\{(Z_\alpha, g_\alpha)\}$ is a maximal totally ordered set. This completes the proof. $\square$

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Sümeyye Çakan
email: `sumeyye.tay@gmail.com`
ORCID: 0000-0001-8761-8564
Department of Mathematics
İnönü University
Malatya 44280
TURKEY

Yılmaz Yılmaz
email: `yyilmaz44@gmail.com`
ORCID: 0000-0002-2197-3579
Department of Mathematics
İnönü University
Malatya 44280
TURKEY

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