Journal of
Mathematics
and Applications

JMA No 41, pp 157-170 (2018)

FG-coupled Fixed Point Theorems for Contractive Type Mappings in Partially Ordered Metric Spaces

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Abstract: In this paper we prove FG-coupled fixed point theorems for Kannan, Reich and Chatterjea type mappings in partially ordered complete metric spaces using mixed monotone property.

AMS Subject Classification: 47H10, 54F05.
Keywords and Phrases: FG-coupled fixed point; Mixed monotone property; Contractive type mappings; Partially ordered space.

1. Introduction and Preliminaries

Banach contraction theorem is one of the fundamental theorems in metric fixed point theory. Banach proved existence of unique fixed point for a self contraction in complete metric space. Since the contractions are always continuous, Kannan introduced a new type of contractive map known as Kannan mapping [8] and proved analogues results of Banach contraction theorem. The importance of Kannan mapping is that it can be discontinuous and it characterizes completeness of the space [14, 15]. In [11] Reich introduced a new type of contraction which is a generalization of Banach contraction and Kannan mapping and proved existence of unique fixed point in complete metric spaces. Later Chatterjea defined a contraction similar to Kannan mapping known as Chatterjea mapping [4] and proved various fixed point results. Inspired by these contractions, several authors did research in this area using different spaces and by weakening the contraction conditions [2, 7, 9, 12].

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [6]. They proved fixed point theorems using mixed monotone property in cone spaces.

In [3] Gnana Bhaskar and Lakshmikantham proved coupled fixed point theorems for contractions in partially ordered complete metric spaces using mixed monotone property. Kannan, Chatterjea and Reich type contractions are further explored in coupled fixed point theory and the results are reported in [1, 5, 13]. Recently the concept of FG-coupled fixed point was introduced in [10] and they proved FG-coupled fixed point theorems for various contractive type mappings.

In this paper we prove existence of FG-coupled fixed point theorems using Kannan, Chatterjea and Reich type contraction on partially ordered complete metric spaces.

Now we recall some basic concepts of coupled and FG-coupled fixed points.

Definition 1.1 ([3]). An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the map $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2 ([10]). Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$. We say that F and G have mixed monotone property if for any $x, y \in X$
$x_1, x_2 \in X,\ \ x_1 \leq_{P_1} x_2 \Rightarrow F(x_1, y) \leq_{P_1} F(x_2, y)$ and $G(y, x_1) \geq_{P_2} G(y, x_2)$
$y_1, y_2 \in Y,\ \ y_1 \leq_{P_2} y_2 \Rightarrow F(x, y_1) \geq_{P_1} F(x, y_2)$ and $G(y_1, x) \leq_{P_2} G(y_2, x)$.

Definition 1.3 ([10]). An element $(x, y) \in X \times Y$ is said to be FG-coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.

If $(x, y) \in X \times Y$ is an FG-coupled fixed point then $(y, x) \in Y \times X$ is a GF-coupled fixed point. Partial order $\leq$ on $X \times Y$ is defined as $(u, v) \leq (x, y) \Leftrightarrow x \geq_{P_1} u,\ y \leq_{P_2} v\ \forall (x, y), (u, v) \in X \times Y$. Also the iteration is given by $F^{n+1}(x, y) = F(F^n(x, y), G^n(y, x))$ and $G^{n+1}(y, x) = G(G^n(y, x), F^n(x, y))$ for every $n \in \mathbb{N}$ and $(x, y) \in X \times Y$.

2. Main Results

Theorem 2.1. *Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying*

$$d_X(F(x, y), F(u, v)) \leq p\ d_X(x, F(x, y)) + q\ d_X(u, F(u, v)); \forall x \geq_{P_1} u,\ y \leq_{P_2} v \quad (1)$$

$$d_Y(G(y, x), G(v, u)) \leq r\ d_Y(y, G(y, x)) + s\ d_Y(v, G(v, u)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (2)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Given $x_0 \leq_{P_1} F(x_0, y_0) = x_1$ and $y_0 \geq_{P_2} G(y_0, x_0) = y_1$.
Define $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ for $n = 1, 2, 3..$
Then we can easily show that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y.
Using inequalities (1) and (2) we get

$$\begin{aligned}
d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\leq p\ d_X(x_n, F(x_n, y_n)) + q\ d_X(x_{n-1}, F(x_{n-1}, y_{n-1})) \\
&= p\ d_X(x_n, x_{n+1}) + q\ d_X(x_{n-1}, x_n) \\
\text{ie, } (1-p)\ d_X(x_{n+1}, x_n) &\leq q\ d_X(x_{n-1}, x_n) \\
\text{ie, } d_X(x_n, x_{n+1}) &\leq \frac{q}{1-p} d_X(x_{n-1}, x_n) \\
&= \delta_1\ d_X(x_{n-1}, x_n) \text{ where } \ \delta_1 = \frac{q}{1-p} < 1 \\
&\leq \delta_1^2\ d_X(x_{n-2}, x_{n-1}) \\
&\quad\vdots \\
&\leq \delta_1^n\ d_X(x_0, x_1).
\end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \leq {\delta_2}^n\ d_Y(y_1, y_0)$ where $\delta_2 = \dfrac{r}{1-s} < 1$.
Consider $m > n$

$$\begin{aligned}
d_X(x_m, x_n) &\leq d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + ... + d_X(x_{n+1}, x_n) \\
&\leq {\delta_1}^{m-1}\ d_X(x_1, x_0) + {\delta_1}^{m-2}\ d_X(x_1, x_0) + ... + {\delta_1}^n\ d_X(x_1, x_0) \\
&= {\delta_1}^n \Big(1 + \delta_1 + ... + {\delta_1}^{m-n-1}\Big)\ d_X(x_1, x_0) \\
&\leqslant \frac{{\delta_1}^n}{1-\delta_1}\ d_X(x_1, x_0).
\end{aligned}$$

Since $0 \leq \delta_1 < 1$, ${\delta_1}^n$ converges to 0(as $n \to \infty$). Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X. Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y. Since by the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
Now we have to prove the existence of FG-coupled fixed point.
Consider,

$$\begin{aligned}
d_X(F(x, y), x) &= \lim_{n\to\infty} d_X(F(F^n(x_0, y_0), G^n(y_0, x_0)), F^n(x_0, y_0)) \\
&= \lim_{n\to\infty} d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \\
&= 0
\end{aligned}$$

ie, $F(x, y) = x$. Similarly we get $G(y, x) = y$. □

By replacing the continuity of F and G by other conditions we obtain the following existence theorems of FG-coupled fixed point.

Theorem 2.2. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq_{P_1} x$ $\forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \leq_{P_2} y_n$ $\forall n$.

Also assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x,y), F(u,v)) \leq p\ d_X(x, F(x,y)) + q\ d_X(u, F(u,v));\ \forall x \geq_{P_1} u,\ y \leq_{P_2} v \quad (3)$$

$$d_Y(G(y,x), G(v,u)) \leq r\ d_Y(y, G(y,x)) + s\ d_Y(v, G(v,u));\ \forall x \leq_{P_1} u,\ y \geq_{P_2} v. \quad (4)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. Following as in the proof of Theorem 2.1 we get $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
Now we have

$$\begin{aligned}
d_X(F(x,y), x) &\leq d_X(F(x,y), F^{n+1}(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0), x) \\
&= d_X(F(x,y), F(F^n(x_0,y_0), G^n(y_0,x_0)) + d_X(F^{n+1}(x_0,y_0), x) \\
&\leq p\ d_X(x, F(x,y)) + q\ d_X(F^n(x_0,y_0), F(F^n(x_0,y_0), G^n(y_0,x_0))) \\
&\quad + d_X(F^{n+1}(x_0,y_0), x) \quad \text{(using (3))}
\end{aligned}$$

ie, $d_X(F(x,y), x) \leq p\ d_X(x, F(x,y))$ as $n \to \infty$.
This holds only when $d_X(F(x,y), x) = 0$. Therefore we get $F(x,y) = x$.
Similarly using (4) and $\lim_{n\to\infty} G^n(y_0, x_0) = y$ we can prove $y = G(y,x)$. □

Remark 2.1. If we put $k = m$ and $l = n$ in Theorems 2.1 and 2.2, we get Theorems 2.7 and 2.8 respectively of [10].

Theorem 2.3. *Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying*

$$d_X(F(x,y), F(u,v)) \leq p\ d_X(x, F(u,v)) + q\ d_X(u, F(x,y));\ \forall\ x \geq_{P_1} u,\ y \leq_{P_2} v \quad (5)$$

$$d_Y(G(y,x), G(v,u)) \leq r\ d_Y(y, G(v,u)) + s\ d_Y(v, G(y,x));\ \forall\ x \leq_{P_1} u,\ y \geq_{P_2} v. \quad (6)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. As in Theorem 2.1 we have $\{x_n\}$ increasing in X and $\{y_n\}$ decreasing in Y. We have

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\le p\ d_X(x_n, F(x_{n-1}, y_{n-1})) + q\ d_X(x_{n-1}, F(x_n, y_n)) \quad \text{(Using (5))} \\ &= p\ d_X(x_n, x_n) + q\ d_X(x_{n-1}, x_{n+1}) \\ &\le q\ [d_X(x_{n-1}, x_n) + d_X(x_n, x_{n+1})] \\ \text{ie,}\ \ d_X(x_n, x_{n+1}) &\le \frac{q}{1-q}\ d_X(x_{n-1}, x_n) \\ &= \delta_1\ d_X(x_{n-1}, x_n) \quad \text{where} \quad \delta_1 = \frac{q}{1-q} < 1 \\ &\le \delta_1^2\ d_X(x_{n-2}, x_{n-1}) \\ &\ \ \vdots \\ &\le \delta_1^n\ d_X(x_0, x_1). \end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \le {\delta_2}^n d_Y(y_1, y_0)$ where $\delta_2 = \dfrac{r}{1-r} < 1$
Now, we prove that $\{F^n(x_0, y_0)\}$ and $\{G^n(y_0, x_0)\}$ are Cauchy sequences in X and Y respectively.
For $m > n$,

$$\begin{aligned} d_X(x_m, x_n) &\le d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + ... + d_X(x_{n+1}, x_n) \\ &\le {\delta_1}^{m-1}\ d_X(x_1, x_0) + {\delta_1}^{m-2}\ d_X(x_1, x_0) + ... + {\delta_1}^n\ d_X(x_1, x_0) \\ &\le \frac{{\delta_1}^n}{1-\delta_1}\ d_X(x_1, x_0). \end{aligned}$$

Since $0 \le \delta_1 < 1$, ${\delta_1}^n$ converges to 0 (as $n \to \infty$). Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X.
Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.
By the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
As in the proof of Theorem 2.1 we can show that $x = F(x, y)$ and $y = G(y, x)$. □

Theorem 2.4. *Let $(X, d_X, \le_{P_1})$ and $(Y, d_Y, \le_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \le_{P_1} x\ \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \le_{P_2} y_n\ \forall n$.

Also assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \le p\ d_X(x, F(u, v)) + q\ d_X(u, F(x, y));\ \forall x \ge_{P_1} u,\ y \le_{P_2} v \quad (7)$$

$$d_Y(G(y,x),G(v,u)) \leq r\ d_Y(y,G(v,u)) + s\ d_Y(v,G(y,x));\ \forall x \leq_{P_1} u,\ y \geq_{P_2} v. \quad (8)$$

If there exist $x_0 \in X,\ y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0,y_0)$ and $y_0 \geq_{P_2} G(y_0,x_0)$ then there exist $x \in X,\ y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. Following as in the proof of Theorem 2.3 we get $\lim_{n\to\infty} F^n(x_0,y_0) = x$ and $\lim_{n\to\infty} G^n(y_0,x_0) = y$.
Consider

$$\begin{aligned}
d_X(F(x,y),x) &\leq d_X(F(x,y),F^{n+1}(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0),x)\\
&= d_X(F(x,y),F(F^n(x_0,y_0),G^n(y_0,x_0))) + d_X(F^{n+1}(x_0,y_0),x)\\
&\leq p\ d_X(x,F((F^n(x_0,y_0),G^n(y_0,x_0))) + q\ d_X(F^n(x_0,y_0),F(x,y))\\
&\quad + d_X(F^{n+1}(x_0,y_0),x)\\
&= p\ d_X(x,F^{n+1}(x_0,y_0)) + q\ d_X(F^n(x_0,y_0),F(x,y))\\
&\quad + d_X(F^{n+1}(x_0,y_0),x)
\end{aligned}$$

ie, $d_X(F(x,y),x) \leq q\,d_X(x,F(x,y))$ as $n \to \infty$, which implies that $d_X(F(x,y),x) = 0$. Therefore we get $F(x,y) = x$.
Similarly using (8) and $\lim_{n\to\infty} G^n(y_0,x_0) = y$, we get $y = G(y,x)$. □

Remark 2.2. If we put $p = r$ and $q = s$ in Theorems 2.3 and 2.4, we get Theorems 2.9 and 2.10 respectively of [10].

The following example illustrates the above results.

Example 2.1. Let $X = [0,1]$ and $Y = [-1,1]$ with usual metric. Partial order on X is defined as $x \leq_{P_1} u$ if and only if $x = u$ and partial order on Y is defined as $y \leq_{P_2} v$ if and only if either $y = v$ or $(y,v) = (0,1)$. The mapping $F : X \times Y \to X$ is defined by $F(x,y) = \dfrac{x+1}{2}$ and $G : Y \times X \to Y$ is defined as $G(y,x) = \dfrac{x-1}{2}$. Then F and G satisfies (1), (2), (5), (6) with $p,q,r,s \in [0,\frac{1}{2})$. Also $(1,0)$ is the FG-coupled fixed point.

Theorem 2.5. *Let $(X,d_X,\leq_{P_1}),(Y,d_Y,\leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that there exist a,b,c with $a+b+c < 1$ satisfying*

$$\begin{aligned}
d_X(F(x,y),F(u,v)) \leq a\ d_X(x,F(x,y)) + b\ d_X(u,F(u,v)) + c\ d_X(x,u);\\
\forall x \geq_{P_1} u,\ y \leq_{P_2} v
\end{aligned} \quad (9)$$

$$\begin{aligned}
d_Y(G(y,x),G(v,u)) \leq a\ d_Y(y,G(y,x)) + b\ d_Y(v,G(v,u)) + c\ d_Y(y,v);\\
\forall x \leq_{P_1} u,\ y \geq_{P_2} v.
\end{aligned} \quad (10)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0,y_0)$ and $y_0 \geq_{P_2} G(y_0,x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. Following as in Theorem 2.1 we have $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y.
Now we claim that

$$d_X(F^{n+1}(x_0,y_0),F^n(x_0,y_0)) \leq \Big(\frac{b+c}{1-a}\Big)^n \ d_X(x_0,x_1) \tag{11}$$

$$d_Y(G^{n+1}(y_0,x_0),G^n(y_0,x_0)) \leq \Big(\frac{a+c}{1-b}\Big)^n \ d_Y(y_0,y_1). \tag{12}$$

The proof is by mathematical induction with the help of (9) and (10).
For $n=1$, consider

$$\begin{aligned} d_X(F^2(x_0,y_0),F(x_0,y_0)) &= d_X(F(F(x_0,y_0),G(y_0,x_0)),F(x_0,y_0)) \\ &\leq a\ d_X(F(x_0,y_0),F^2(x_0,y_0)) + b\ d_X(x_0,F(x_0,y_0)) \\ &\quad + c\ d_X(F(x_0,y_0),x_0) \\ \text{ie, } d_X(F^2(x_0,y_0),F(x_0,y_0)) &\leq \frac{b+c}{1-a}\ d_X(x_0,x_1). \end{aligned}$$

Thus the inequality (11) is true for $n=1$.
Now assume that (11) is true for $n \leq m$, and check for $n=m+1$.
Consider,

$$\begin{aligned} &d_X(F^{m+2}(x_0,y_0),F^{m+1}(x_0,y_0)) \\ &\quad = d_X(F(F^{m+1}(x_0,y_0),G^{m+1}(y_0,x_0)),F(F^m(x_0,y_0),G^m(y_0,x_0))) \\ &\quad \leq a\ d_X(F^{m+1}(x_0,y_0),F^{m+2}(x_0,y_0)) + b\ d_X(F^m(x_0,y_0),F^{m+1}(x_0,y_0)) \\ &\quad + c\ d_X(F^{m+1}(x_0,y_0),F^m(x_0,y_0)) \end{aligned}$$

$$\begin{aligned} \text{ie, } d_X(F^{m+2}(x_0,y_0),F^{m+1}(x_0,y_0)) &\leq \frac{b+c}{1-a}\ d_X(F^m(x_0,y_0),F^{m+1}(x_0,y_0)) \\ &\leq \Big(\frac{b+c}{1-a}\Big)^{m+1}\ d_X(x_0,x_1) \end{aligned}$$

ie, the inequality (11) is true for all $n \in \mathbb{N}$.
Similarly we can prove the inequality (12).
For $m > n$, consider

$$\begin{aligned} &d_X(F^n(x_0,y_0),F^m(x_0,y_0)) \\ &\quad \leq d_X(F^n(x_0,y_0),F^{n+1}(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0),F^{n+2}(x_0,y_0)) + ... \\ &\quad + d_X(F^{m-1}(x_0,y_0),F^m(x_0,y_0)) \\ &\quad \leq \left[\Big(\frac{b+c}{1-a}\Big)^n + \Big(\frac{b+c}{1-a}\Big)^{n+1} + ... + \Big(\frac{b+c}{1-a}\Big)^{m-1}\right] d_X(x_0,x_1) \\ &\quad \leq \frac{{\delta_1}^n}{1-\delta_1}\ d_X(x_0,x_1) \text{ where } \delta_1 = \frac{b+c}{1-a} < 1. \end{aligned}$$

Since $0 \leq \delta_1 < 1$, ${\delta_1}^n$ converges to 0 (as $n \to \infty$) ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X. Similarly by using inequality (12) we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.
By the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
As in the proof of Theorem 2.1, using continuity of F and G we can prove that $F(x, y) = x$ and $G(y, x) = y$. □

If we take $X = Y$ and $F = G$ in the above theorem we get the following corollary.

Corollary 2.1. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a continuous function having the mixed monotone property. Assume that there exist non-negative a, b, c such that $a + b + c < 1$ satisfying*

$$d(F(x, y), F(u, v)) \leq a\ d(x, F(x, y)) + b\ d(u, F(u, v)) + c\ d(x, u);\ \forall x \geq u,\ y \leq v.$$

If there exist $x_0, y_0 \in X$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.6. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq_{P_1} x\ \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \leq_{P_2} y_n\ \forall n$.

Also asuume that there exist a, b, c with $a + b + c < 1$ satisfying

$$d_X(F(x, y), F(u, v)) \leq a\ d_X(x, F(x, y)) + b\ d_X(u, F(u, v)) + c\ d_X(x, u); \quad \forall x \geq_{p_1} u,\ y \leq_{p_2} v \tag{13}$$

$$d_Y(G(y, x), G(v, u)) \leq a\ d_Y(y, G(y, x)) + b\ d_Y(v, G(v, u)) + c\ d_Y(y, v); \quad \forall x \leq_{P_1} u,\ y \geq_{P_2} v. \tag{14}$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.5 we obtain $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
We have

$$\begin{aligned}
d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x)\\
&= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x)\\
&\leq a\ d_X(x, F(x, y)) + b\ d_X(F^n(x_0, y_0), F(F^n(x_0, y_0), G^n(y_0, x_0)))\\
&\quad + c\ d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x)\\
&= a\ d_X(x, F(x, y)) + b\ d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0))\\
&\quad + c\ d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x)
\end{aligned}$$

ie, $d_X(F(x,y),x) \leq a\, d_X(x,F(x,y))$ as $n \to \infty$, which implies that $d_X(F(x,y),x) = 0$. Therefore $F(x,y) = x$.
Similarly using (14) and $\lim_{n\to\infty} G^n(y_0,x_0) = y$ we get $y = G(y,x)$. □

By assuming $X = Y$ and $F = G$ in the above theorem we will get the following corollary.

Corollary 2.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property. Assume that X satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ $\forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n$ $\forall n$.

Also assume that there exist non-negative a, b, c such that $a + b + c < 1$ satisfying

$$d(F(x,y),F(u,v)) \leq a\ d(x,F(x,y)) + b\ d(u,F(u,v)) + c\ d(x,u);\ \forall x \geq u,\ y \leq v.$$

If there exist $(x_0,y_0) \in X \times X$ satisfying $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$ then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Remark 2.3. If we take $c = 0$ in Theorems 2.5 and 2.6, we get Theorems 2.7 and 2.8 respectively of [10].

Theorem 2.7. *Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that there exist non-negative a,b,c satisfying*

$$\begin{aligned} d_X(F(x,y),F(u,v)) \leq a\ d_X(x,F(u,v)) + b\ d_X(u,F(x,y)) + c\ d_X(x,u);& \\ \forall x \geq_{P_1} u,\ y \leq_{P_2} v;\ 2b + c < 1& \end{aligned} \tag{15}$$

$$\begin{aligned} d_Y(G(y,x),G(v,u)) \leq a\ d_Y(y,G(v,u)) + b\ d_Y(v,G(y,x)) + c\ d_Y(y,v);& \\ \forall x \leq_{P_1} u,\ y \geq_{P2} v;\ 2a + c < 1.& \end{aligned} \tag{16}$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0,y_0)$ and $y_0 \geq_{P_2} G(y_0,x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. As in the proof of Theorem 2.1, it can be proved that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y.
Now we claim that

$$d_X(F^{n+1}(x_0,y_0),F^n(x_0,y_0)) \leq \left(\frac{b+c}{1-b}\right)^n d_X(x_0,x_1) \tag{17}$$

$$d_Y(G^{n+1}(y_0,x_0),G^n(y_0,x_0)) \leq \left(\frac{a+c}{1-a}\right)^n d_Y(y_0,y_1). \tag{18}$$

We prove the claim by mathematical induction, using (15) and (16).
For $n = 1$, consider

$$\begin{aligned}
&d_X(F^2(x_0,\, y_0), F(x_0, y_0)) \\
&\qquad = d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\
&\qquad \le a\ d_X(F(x_0, y_0), F(x_0, y_0)) + b\ d_X(x_0, F^2(x_0, y_0)) + c\ d_X(F(x_0, y_0), x_0) \\
&\qquad \le b\ [d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0))] + c\ d_X(F(x_0, y_0), x_0)
\end{aligned}$$

ie, $d_X(F^2(x_0, y_0), F(x_0, y_0)) \le \dfrac{b+c}{1-b}\ d_X(x_0, x_1)$.
Thus the inequality (17) is true for $n = 1$.
Now assume that (17) is true for $n \le m$, then check for $n = m + 1$.
Consider,

$$\begin{aligned}
&d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \\
&\qquad = d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\
&\qquad \le a\ d_X(F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)) + b\ d_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \\
&\qquad + c\ d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\
&\qquad \le b\ [d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0))] \\
&\qquad + c\ d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0))
\end{aligned}$$

ie,

$$\begin{aligned}
d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &\le \frac{b+c}{1-b}\ d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \\
&\le \Big(\frac{b+c}{1-b}\Big)^{m+1}\ d_X(x_0, x_1)
\end{aligned}$$

ie, the inequality (17) is true for all $n \in \mathbb{N}$.
Similarly we can prove the inequality (18).
For $m > n$, consider

$$\begin{aligned}
&d_X(F^n(x_0,\, y_0), F^m(x_0, y_0)) \\
&\qquad \le d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)) + ... \\
&\qquad + d_X(F^{m-1}(x_0, y_0), F^m(x_0, y_0)) \\
&\qquad \le \Bigg[\Big(\frac{b+c}{1-b}\Big)^n + \Big(\frac{b+c}{1-b}\Big)^{n+1} + ... + \Big(\frac{b+c}{1-b}\Big)^{m-1}\Bigg]\ d_X(x_0, x_1) \\
&\qquad \le \frac{{\delta_1}^n}{1-\delta_1}\ d_X(x_0, x_1);\ \text{where } \delta_1 = \frac{b+c}{1-b} < 1.
\end{aligned}$$

Since $0 \le \delta_1 < 1$, ${\delta_1}^n$ converges to 0 (as $n \to \infty$) ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X. Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.

Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
By continuity of F and G, as in the Theorem 2.1 we can show that $F(x, y) = x$ and $G(y, x) = y$. □

If $X = Y$ and $F = G$ in the above theorem we get the following corollary.

Corollary 2.3. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a continuous function having the mixed monotone property. Assume that there exist non-negative a, b, c such that $2a+c < 1$ and $2b+c < 1$ satisfying*

$$d(F(x,y), F(u,v)) \leq a\ d(x, F(u,v)) + b\ d(u, F(x,y)) + c\ d(x,u);\ \forall\ x \geq u,\ \ y \leq v.$$

If there exist $(x_0, y_0) \in X \times Y$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

In the following theorem we replace the continuity by other conditions to obtain FG-coupled fixed point.

Theorem 2.8. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq_{P_1} x\ \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \leq_{P_2} y_n\ \forall n$.

Also assume that there exist non-negative a,b,c satisfying

$$\begin{aligned} d_X(F(x,y), F(u,v)) \leq a\ d_X(x, F(u,v)) &+ b\ d_X(u, F(x,y)) + c\ d_X(x,u); \\ &\forall x \geq_{P_1} u,\ \ y \leq_{P_2} v;\ \ 2b + c < 1 \end{aligned} \tag{19}$$

$$\begin{aligned} d_Y(G(y,x), G(v,u)) \leq a\ d_Y(y, G(v,u)) &+ b\ d_Y(v, G(y,x)) + c\ d_Y(y,v); \\ &\forall x \leq_{P_1} u,\ \ y \geq_{P_2} v;\ \ 2a + c < 1. \end{aligned} \tag{20}$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.7 we get $\lim_{n\to\infty} F^n(x_0, y_0) = x$ and $\lim_{n\to\infty} G^n(y_0, x_0) = y$.
We have

$$\begin{aligned} d_X(F(x,y), x) &\leq d_X(F(x,y), F^{n+1}(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0), x) \\ &= d_X(F(x,y), F(F^n(x_0,y_0), G^n(y_0,x_0)) + d_X(F^{n+1}(x_0,y_0), x) \\ &\leq a\ d_X(x, F(F^n(x_0,y_0), G^n(y_0,x_0))) + b\ d_X(F^n(x_0,y_0), F(x,y)) \\ &\quad + c\ d_X(x, F^n(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0), x) \\ &= a\ d_X(x, F^{n+1}(x_0,y_0)) + b\ d_X(F^n(x_0,y_0), F(x,y)) \\ &\quad + c\ d_X(x, F^n(x_0,y_0)) + d_X(F^{n+1}(x_0,y_0), x) \end{aligned}$$

ie, $d_X(F(x,y),x) \leq b\, d_X(x, F(x,y))$ as $n \to \infty$, which implies that $d_X(F(x,y),x) = 0$. Therefore $F(x,y) = x$.
Also by using (20) and $\lim_{n\to\infty} G^n(y_0, x_0) = y$ we can show that $y = G(y,x)$. □

Taking $X = Y$ and $F = G$ in the above corollary we get the corresponding coupled fixed point result.

Corollary 2.4. *Let $(X, d, \leq)$ be a partially ordered complete metric spaces and $F : X \times Y \to X$ be a mapping having the mixed monotone property. Assume that X satisfy the following property*

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x\ \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n\ \forall n$.

Also assume that there exist non-negative a, b, c such that $2a + c < 1$ and $2b + c < 1$ satisfying

$$d(F(x,y), F(u,v)) \leq a\ d(x, F(u,v)) + b\ d(u, F(x,y)) + c\ d(x,u); \forall x \geq u,\ y \leq v.$$

If there exist $(x_0, y_0) \in X \times Y$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $(x, y) \in X \times Y$ such that $x = F(x,y)$ and $y = F(y,x)$.

Remark 2.4. If we take $c = 0$ in Theorems 2.7 and 2.8, we get Theorems 2.9 and 2.10 respectively of [10].

Acknowledgment

The first author acknowledges financial support from Kerala State Council for Science, Technology and Environment (KSCSTE), in the form of fellowship. We also acknowledge the valuable suggestions made by the referee for improving this paper.

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DOI: 10.7862/rf.2018.11

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Received 05.01.2018 *Accepted 10.08.2018*