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Number of Zeros of a Polynomial (Lacunary-type) in a Disk

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Abstract: The problem of finding out the region which contains all or a prescribed number of zeros of a polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ has a long history and dates back to the earliest days when the geometrical representation of complex numbers was introduced. In this paper, we present certain results concerning the location of the zeros of Lacunary-type polynomials $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j$ in a disc centered at the origin.

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1. Introduction and Statement of Results

The problem of locating some or all the zeros of a given polynomial as a function of its coefficients is of long standing interest in mathematics. This fact can be visualized by glancing at the references in the comprehensive books of Marden [9] and Milovanovic, Mitrinovic and Rassias [10], Rahman and Schmeisser [12] and by noting the abundance of recent publications on the subject [7, 8, 13].

Regarding the least number of zeros of polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ in a given circle Mohammad [11] proved the following:

Theorem A. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [3] generalized Theorem A to polynomials with complex coefficients and proved the following result:

Theorem B. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j,\ j = 0, 1, 2, \ldots, n$ *such that*

$$\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

In this direction, recently Irshad et al [1] proved the following:

Theorem C. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some* $\lambda \geq 1$, $0 \leq k \leq n$,

$$|a_n| \leq |a_{n-1}| \leq \ldots \leq |a_{k+1}| \leq \lambda |a_k| \geq |a_{k-1}| \geq \ldots \geq |a_1| \geq |a_0|,$$

and for some real β,

$$|\arg a_j - \beta| \leq \alpha \leq \tfrac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \left\{ \frac{2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha}{|a_0|} + \frac{|a_n|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| + 2|1-\lambda||a_k|}{|a_0|} \right\}.$$

Chan and Malik [2] introduced the class of Lacunary polynomials of the form $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j$, where $a_0 \neq 0$. Notice that when $\mu = 1$, we simply have the class of all polynomials of degree n. In [5] and [6] Landau proved that every trinomial

$$a_0 + a_1 z + a_n z^n, a_1 a_n \neq 0, n \geq 2$$

has at least one zero in the circle $|z| \leq 2|\frac{a_0}{a_1}|$ and that of quadrinomial

$$a_0 + a_1 z + a_m z^m + a_n z^n, a_1 a_m a_n \neq 0, 2 \leq m \leq n$$

has at least one zero in the circle $|z| \leq \frac{17}{3}|\frac{a_0}{a_1}|$. These two polynomials are of the Lacunary-type $P(z) := a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$.

The aim of this paper is to study the number of zeros in a disc centered at the origin for such class of polynomials. We begin by proving the following result putting restrictions on the moduli of the coefficients. In fact we prove:

Theorem 1. *Let* $P(z) := a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n-1,\ a_0 \neq 0,$ *be a polynomial of degree* n*. If for some real* α *and* β

$$|\arg a_j - \beta| \leq \ \alpha \leq \ \frac{\pi}{2}, \qquad \mu \leq j \leq n$$

and for some $t > 0$ *and some* k *with* $\mu \leq k \leq n$,

$$t^{\mu}|a_{\mu}| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n|a_n|$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0|t + |a_{\mu}|t^{\mu+1}(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + |a_n|t^{n+1}(1 - \sin\alpha - \cos\alpha) + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha.$$

For $t = 1$, we get the following:

Corollary 1.1. *Let* $P(z) := a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n-1,\ a_0 \neq 0$ *be a polynomial of degree* n*. If for some real* α *and* β

$$|\arg a_j - \beta| \leq \ \alpha \leq \ \frac{\pi}{2}, \qquad \mu \leq j \leq n$$

and some k *with*

$$|a_{\mu}| \leq \cdots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_{\mu}|(1 - \sin\alpha - \cos\alpha) +2|a_k|\cos\alpha + |a_n|(1 - \sin\alpha - \cos\alpha) + 2\sum_{j=\mu}^{n} |a_j|\sin\alpha.$$

With $k = n$ in Corollary 1.1, we get:

Corollary 1.2. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n-1,\ a_0 \ne 0$ *be a polynomial of degree* n. *If for some real* α *and* β

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \qquad \mu \le j \le n$$

such that

$$|a_\mu| \le \cdots \le |a_{n-1}| \le |a_n|$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_\mu|(1 - \sin\alpha - \cos\alpha) + |a_n|(1 - \sin\alpha + \cos\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha.$$

Choosing $k = \mu$ in Corollary 1.1, we get:

Corollary 1.3. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n-1,\ a_0 \ne 0$ *be a polynomial of degree* n. *If for some real* α *and* β

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \qquad \mu \le j \le n$$

such that

$$|a_\mu| \ge \cdots \ge |a_{n-1}| \ge |a_n|$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_\mu|(1 - \sin\alpha + \cos\alpha) + |a_n|(1 - \sin\alpha - \cos\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha.$$

Taking $\mu = 1$ in Corollary 1.3, we have

Corollary 1.4. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j,\ a_0 \ne 0$ *be a polynomial of degree* n. *If for some real* α *and* β

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \qquad \mu \le j \le n$$

such that

$$|a_1| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_1|(1 - \sin\alpha + \cos\alpha) + |a_n|(1 - \sin\alpha - \cos\alpha) + 2\sum_{j=1}^{n} |a_j| \sin\alpha.$$

Next, we put restriction on the real part of coefficients of a polynomial and proved:

Theorem 2. *Let* $P(z) := a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n-1,\ a_0 \neq 0,$ *be a polynomial of degree* n *with Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \leq k \leq n$ *we have*

$$t^{\mu}\alpha_{\mu} \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_n$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j| t^{j+1}.$$

For $t = 1$ in Theorem 2, we obtain

Corollary 2.1. *Let* $P(z) := a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n-1,\ a_0 \neq 0,$ *be a polynomial of degree* n *with Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* k *with* $\mu \leq k \leq n$ *we have*

$$\alpha_{\mu} \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|.$$

For $k = n$ in Corollary 2.1, we get:

Corollary 2.2. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n-1,\ a_0 \ne 0,$ *be a polynomial of degree* n *with* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \le \cdots \le \alpha_{n-1} \le \alpha_n$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\sum_{j=\mu}^{n} |\beta_j|.$$

For $k = \mu$, in Corollary 2.1, we get:

Corollary 2.3. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n-1,\ a_0 \ne 0,$ *be a polynomial of degree* n *with* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \ge \cdots \ge \alpha_{n-1} \ge \alpha_n$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu) + (|\alpha_\mu| - \alpha_\mu) + 2\sum_{j=\mu}^{n} |\beta_j|.$$

For $\beta_j = 0$, $1 \le j \le n$ in Theorem 2, we have the following:

Corollary 2.4. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j$, $\le \mu \le n-1$, *where* $a_0 \ne 0$. *Suppose that for some* $t > 0$ *and some* k *we have*

$$t^\mu a_\mu \le \cdots \le t^{k-1} a_{k-1} \le t^k a_k \ge t^{k+1} a_{k+1} \ge \cdots \ge t^{n-1} a_{n-1} \ge t^n a_n$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0|t + (|a_\mu| - a_\mu)t^{\mu+1} + 2a_k t^{k+1} + (|a_n| - a_n)t^{n+1}.$$

Finally, we prove the following result:

Theorem 3. *Let* $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n-1$, *where* $a_0 \ne 0$, *Re* $a_j = \alpha_j$ *and* *Im* $a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$ *we have*

$$t^\mu \alpha_\mu \le \cdots \le t^{k-1}\alpha_{k-1} \le t^k \alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots \ge t^{n-1}\alpha_{n-1} \ge t^n \alpha_n$$

and for some $\mu \le l \le n$ *we have*

$$t^\mu \beta_\mu \le \cdots \le t^{l-1}\beta_{l-1} \le t^l \beta_l \ge t^{l+1}\beta_{l+1} \ge \cdots \ge t^{n-1}\beta_{n-1} \ge t^n \beta_n$$

then the number of zeros of $P(z)$ *in* $|z| \le \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1}$$
$$+ 2(\alpha_k t^{k+1} + \beta_l t^{l+1})t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}.$$

If we take $t = 1$, in Theorem 3 we obtain:

Corollary 3.1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n-1$, *where* $a_0 \ne 0$, *Re* $a_j = \alpha_j$ *and* *Im* $a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* k *with* $\mu \le k \le n$ *we have*

$$\alpha_\mu \le \cdots \le \alpha_{k-1} \le \alpha_k \ge \alpha_{k+1} \ge \cdots \ge \alpha_{n-1} \ge \alpha_n$$

and for some $\mu \le l \le n$ *we have*

$$\beta_\mu \le \cdots \le \beta_{l-1} \le \beta_l \ge \beta_{l+1} \ge \cdots \ge \beta_{n-1} \ge \beta_n.$$

Then the number of zeros of $P(z)$ *in* $|z| \le \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_l) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

For $k = l = n$ in Corollary 3.1, we get the following:

Corollary 3.2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n-1$, *where* $a_0 \ne 0$, *Re* $a_j = \alpha_j$ *and* *Im* $a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \le \cdots \le \alpha_{n-1} \le \alpha_n$$

and

$$\beta_\mu \leq \cdots \leq \beta_{n-1} \leq \beta_n$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n).$$

In Corollary 3.1, if we choose $k = l = \mu$ we get:

Corollary 3.3. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n-1,$ *where* $a_0 \neq 0,$ *Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$ *such that*

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \leq \alpha_n$$

and

$$\beta_\mu \geq \cdots \geq \beta_{n-1} \geq \beta_n$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

2. Lemma

For the proof of some these results we need the following lemma which is due to Govil and Rahman [4].

Lemma 2.1. *For any two complex numbers* b_0 *and* b_1 *such that* $|b_0| \geq |b_1|$ *and*

$$|arg\ \ b_j - \beta| \leq\ \alpha \leq\ \frac{\pi}{2}, \quad j = 0, 1$$

for some β, *then*

$$|b_0 - b_1| \leq (|b_0| - |b_1|) \cos \alpha + (|b_0| + |b_1|) \sin \alpha.$$

An application of the Maximum modulus theorem shown in (p.171, [14]) we have the following interesting result:

Lemma 2.2. *Let $f(z)$ be regular and $|f(z)| \leq M$, in the circle $|z| \leq R$ and suppose that $f(0) \neq 0$, then the number of zeros of $f(z)$ in the circle $|z| \leq \frac{1}{2}R$ does not exceed* $\frac{1}{\log 2} \log \left[\frac{M}{|f(0)|}\right]$.

3. Proofs of Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= (t-z)\left(a_0 + \sum_{j=\mu}^{n} a_j z^j\right) \\ &= a_0 t + \sum_{j=\mu}^{n} a_j t z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} \\ &= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\ &= a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned} |F(z)| &\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |a_j t - a_{j-1}|t^j + |a_n|t^{n+1} \\ &= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j t - a_{j-1}|t^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j t|t^j + |a_n|t^{n+1}. \end{aligned}$$

Using Lemma 2.1 with $b_0 = a_j t$ and $b_1 = a_{j-1}$ when $1 \leq j \leq k$ and with $b_0 = a_{j-1}$ and $b_1 = a_j t$ when $k+1 \leq j \leq n$,

$$\begin{aligned} |F(z)| \leq 2|a_0|t + |a_\mu|t^{\mu+1} &+ \sum_{j=\mu+1}^{k} \{(|a_j|t - |a_{j-1}|)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\}t^j \\ &+ \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\}t^j + |a_n|t^{n+1} \end{aligned}$$

$$= 2|a_0|t+|a_\mu|t^{\mu+1}+\sum_{j=\mu+1}^{k}|a_j|t^{j+1}\cos\alpha-\sum_{j=\mu+1}^{k}|a_{j-1}|t^{j}\cos\alpha+\sum_{j=\mu+1}^{k}|a_j|t^{j+1}\sin\alpha$$

$$+\sum_{j=\mu+1}^{k}|a_{j-1}|t^{j}\sin\alpha+\sum_{j=k+1}^{n}|a_{j-1}|t^{j}\cos\alpha-\sum_{j=k+1}^{n}|a_j|t^{j+1}\cos\alpha$$

$$+\sum_{j=k+1}^{n}|a_{j-1}|t^{j}\sin\alpha+\sum_{j=k+1}^{n}|a_j|t^{j+1}\sin\alpha+|a_n|t^{n+1}$$

$$= 2|a_0|t+|a_\mu|t^{\mu+1}-|a_\mu|t^{\mu+1}\cos\alpha+|a_k|t^{k+1}\cos\alpha+|a_\mu|t^{\mu+1}\sin\alpha+|a_k|t^{k+1}\sin\alpha$$

$$+2\sum_{j=\mu+1}^{k-1}|a_j|t^{j+1}\sin\alpha+|a_k|t^{k+1}\cos\alpha-|a_n|t^{n+1}\cos\alpha+|a_k|t^{k+1}\sin\alpha$$

$$+|a_n|t^{n+1}\sin\alpha+2\sum_{j=k+1}^{n-1}|a_j|t^{j+1}\sin\alpha+|a_n|t^{n+1}$$

$$= 2|a_0|t+|a_\mu|t^{\mu+1}+|a_\mu|t^{\mu+1}(\sin\alpha-\cos\alpha)+2\sum_{j=\mu+1}^{n-1}|a_j|t^{j+1}\sin\alpha+2|a_k|t^{k+1}\cos\alpha$$

$$+(\sin\alpha-\cos\alpha+1)|a_n|t^{n+1}$$

$$= 2|a_0|t+|a_\mu|t^{\mu+1}(1-\sin\alpha-\cos\alpha)+2|a_k|t^{k+1}\cos\alpha+|a_n|t^{n+1}(1-\sin\alpha-\cos\alpha)$$

$$+2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha$$

$$= M(say).$$

Now $F(z)$ is analytic in $|z|\le t$ and $F(z)\le M$ for $|z|=t$. Applying Lemma 2.2 to the polynomial $F(z)$, we get the number of zeros of $F(z)$ in $|z|\le \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{M}{|f(0)|}.$$

Thus, the number of zeros of $F(z)$ in $|z|\le\frac{t}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{2|a_0|t+|a_\mu|t^{\mu+1}(1-\sin\alpha-\cos\alpha)+2|a_k|t^{k+1}\cos\alpha}{|a_0|}\right.$$
$$\left.+\frac{|a_n|t^{n+1}(1-\sin\alpha-\cos\alpha)+2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha}{|a_0|}\right\}.$$

As the number of zeros of $P(z)$ in $|z|\le\frac{1}{2}$ is also equal to the number of zeros $F(z)$ the theorem follows. □

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}
F(z) &= (t-z)P(z) \\
&= (t-z)\left(a_0 + \sum_{j=\mu}^{n} a_j z^j\right) \\
&= a_0 t + \sum_{j=\mu}^{n} a_j t z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} \\
&= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j
\end{aligned}$$

and therefore

$$F(z) = (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1})z^j$$

$$+ i \sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.$$

For $|z| = t$, we have

$$|F(z)| \leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j$$

$$+ \sum_{j=\mu+1}^{n} (|\beta_j|t + |\beta_{j-1}|)t^j + (|\alpha_n| + |\beta_n|)t^{n+1}$$

$$= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1})t^j$$

$$+ \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t)t^j + |\beta_\mu|t^{\mu+1} + 2\sum_{j=\mu+1}^{n-1} |\beta_j|t^{j+1} + |\beta_n|t^{n+1}$$

$$+(|\alpha_n| + |\beta_n|)t^{n+1} = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t)t^{k+1}$$

$$-\alpha_n t^{n+1} + |\beta_\mu|t^{\mu+1} + 2\sum_{j=\mu+1}^{n} |\beta_j|t^{j+1} + |\alpha_n|t^{n+1} = 2(|\alpha_0| + |\beta_0|)t$$

$$+(|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j|t^{j+1} = M.$$

Proceedings on the same lines of the proof of Theorem 1, the proof of this result follows.

Proof of Theorem 3. Consider the polynomial

$$F(z) = (t-z)P(z) = a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu) t z^\mu \\ &+ \sum_{j=\mu+1}^{n} ((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1})) z^j - (\alpha_n + i\beta_n) z^{n+1} \\ &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu) t z^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1}) z^j \\ &+ i \sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1}) z^j - (\alpha_n + i\beta_n) z^{n+1}. \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned} |F(z)| &\le 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|) t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}| t^j \\ &+ \sum_{j=\mu+1}^{n} |\beta_j t - \beta_{j-1}| t^j + (|\alpha_n| + |\beta_n|) t^{n+1} = 2(|\alpha_0| + |\beta_0|)t \\ &+ (|\alpha_\mu| + |\beta_\mu|) t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1}) t^j + \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t) t^j \\ &+ \sum_{j=\mu+1}^{l} (\beta_j t - \beta_{j-1}) t^j + \sum_{j=l+1}^{n} (\beta_{j-1} - \beta_j t) t^j + (|\alpha_n| + |\beta_n|) t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|) t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} \\ &- \alpha_n t^{n+1} - \beta_\mu t^{\mu+1} + 2\beta_l t^{l+1} - \beta_n t^{n+1} + (|\alpha_n| + |\beta_n|) t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) t^{\mu+1} \\ &+ 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n) t^{n+1} = M. \end{aligned}$$

The result now follows as in the proof of Theorem 1. □

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