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About a Class of Analytic Functions Defined by Noor-Sălăgean Integral Operator

Olga Engel and Yao Liang Chung

Abstract: In this paper we intoduce a new integral operator as the convolution of the Noor and Sălăgean integral operators. With this integral operator we define the class $C_{NS}(\alpha)$, where $\alpha \in [0, 1)$ and we study some properties of this class.

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1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(U)$ denote the set of holomorphic (analytic) functions in U. We denote by

$$\mathcal{A} = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$$

and

$$S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}.$$

We say that f is starlike in U if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to origin. It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in U.$$

The class of starlike functions with respect to origin is denoted by S^*.

Let T denote a subclass of $\mathcal{A}$ consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

where $a_j \geq 0$, $j = 2, 3, ...$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. For the class T, the followings are equivalent [6]:

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$,

(ii) $f \in T \cap S$,

(iii) $f \in T^*$, where $T^* = T \cap S^*$.

Let

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \; a_j \geq 0, \; j = 2, 3, ...$$

and

$$g(z) = z - \sum_{j=2}^{\infty} b_j z^j, \; b_j \geq 0, \; j = 2, 3, ...$$

then the convolution or the Hadamard product is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \; z \in U.$$

The study of operators plays an important role in geometric function theory. For $f \in H(U)$, $f(0) = 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the I_S^n Sălăgean integral operator is defined as follows [7]:

(i) $I_S^0 f(z) = f(z)$,

(ii) $I_S^1 f(z) = If(z) = \int\limits_0^z f(t) t^{-1} dt$,

(iii) $I_S^n f(z) = I_S(I_S^{n-1} f(z))$.

We remark that if f has the form (1.1), then

$$I_S^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{j^n} z^j, \tag{1.2}$$

where $n \in \mathbb{N}_0$.

In [5] Noor defined an integral operator $I_N^n : \mathcal{A} \to \mathcal{A}$ as follows

$$I_N^n f(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} I_N^n(f(t)) dt, \tag{1.3}$$

where $n \in \mathbb{N}_0$.

Let $f_n(z) = \dfrac{z}{(1-z)^{n+1}}$ and let $f_n^{(-1)}(z)$ be defined such that

$$f_n^{(-1)}(z) * f_n(z) = \frac{z}{1-z}.$$

We note that

$$I_N^n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f(z).$$

We remark that if f has the form (1.1), then

$$I_N^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{C(n,j)} z^j, \tag{1.4}$$

where $C(n,j) = \dfrac{(n+j-1)!}{n!(j-1)!}$.

2. Preliminaries

The following definitions and lemmas will be required in the sequel.

Definition 2.1. [2, 3] Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exist a function w, which is analytic in U and for which $w(0) = 0$, $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. We denote by $\prec$ the subordination relation.

Definition 2.2. [3] Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \longrightarrow \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$.

Lemma 2.1. [2, 3] *Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $p \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \backslash E(q)$ and a number $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,*
(i) $p(z_0) = q(\zeta_0)$
(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$
(iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$.

The following result is a particular case of Lemma 2.1.

Lemma 2.2. [2, 3] *Let $p(z) = 1 + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv 1$ and $n \geq 1$. If $\operatorname{Re} p(z) \not> 0$, $z \in U$, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that*
(i) $p(z_0) = ix$
(ii) $z_0 p'(z_0) = y \leq -\frac{n(x^2+1)}{2}$,
(iii) $\operatorname{Re} z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0$.

If $f(z)=z-\sum_{j=2}^{\infty} a_j z^j$, using the Noor and Sălăgean integral operators we define a new operator as follows:

$$I_{NS}^n f(z)=I_N^n f(z)*I_S^n f(z)=z-\sum_{j=2}^{\infty}\frac{a_j^2}{j^n C(n,j)}z^j, \tag{2.1}$$

where $C(n,j)=\dfrac{(n+j-1)!}{n!(j-1)!}$ and $n\in\mathbb{N}_0$.

Remark 2.1. Differentiate the relation (2.1), we get

$$[I_{NS}^n f(z)]'=1-\sum_{j=2}^{\infty}\frac{a_j^2}{j^{n-1}C(n,j)}z^{j-1}. \tag{2.2}$$

Multiplicating the equality (2.2) with $\dfrac{z}{n}$ we obtain

$$\frac{z}{n}[I_{NS}^n f(z)]'=\frac{z}{n}-\sum_{j=2}^{\infty}\frac{a_j^2}{nj^{n-1}C(n,j)}z^j,$$

which is equivalent to

$$\frac{z}{n}[I_{NS}^n f(z)]'+\frac{z}{n}(n-1)=z-\sum_{j=2}^{\infty}\frac{a_j^2}{nj^{n-1}C(n,j)}z^j. \tag{2.3}$$

Now let $g\in T$ and $g(z)=z-\sum_{j=2}^{\infty}(n+j-1)z^j$. Then from (2.3), we obtain the following relation between $I_{NS}^{n-1}f(z)$ and $I_{NS}^n f(z)$ operators:

$$I_{NS}^{n-1}f(z)=\frac{z}{n}[I_{NS}^n f(z)]'*g(z)+\frac{n-1}{n}z*g(z). \tag{2.4}$$

Using the Noor-Sălăgean integral operator, we define the following class of analytic functions:

Definition 2.3. A function $f\in T$ belongs to the class $C_{NS}(\alpha)$ if

$$\operatorname{Re}\frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)}>\alpha, \tag{2.5}$$

where $\alpha\in[0,1)$ and $z\in U$.

3. Main Results

Theorem 3.1. *Let $f(z)=z-\sum_{j=2}^{\infty}a_j z^j$. Then $f\in C_{NS}(\alpha)$ if and only if*

$$\sum_{j=2}^{\infty}\frac{a_j^2}{j^{n-1}C(n,j)}\left[1-\frac{\alpha}{j}\right]<1-\alpha. \tag{3.1}$$

Proof. Let $f \in C_{NS}(\alpha)$, then we have

$$\operatorname{Re} \frac{z[I^n_{NS} f(z)]'}{I^n_{NS} f(z)} > \alpha, \ z \in U.$$

If $z \in [0, 1)$, we obtain

$$\frac{z - \sum\limits_{j=2}^{\infty} \dfrac{a_j^2}{j^{n-1} C(n,j)} z^j}{z - \sum\limits_{j=2}^{\infty} \dfrac{a_j^2}{j^n C(n,j)} z^j} > \alpha. \tag{3.2}$$

Since the denominator of (3.2) is positive, the relation (3.2) is equivalent with

$$\alpha - 1 < \sum_{j=2}^{\infty} \Big[\frac{\alpha a_j^2}{j^n C(n,j)} z^{j-1} - \frac{a_j^2}{j^{n-1} C(n,j)} z^{j-1} \Big],$$

and finally we get

$$\alpha - 1 < \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} z^{j-1} \Big[\frac{\alpha}{j} - 1 \Big].$$

Considering $z \to 1^-$ along to the real axis, we get:

$$\alpha - 1 < \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} \Big[\frac{\alpha}{j} - 1 \Big].$$

To prove the reciproc implication we consider f with the form (1.1) and for which the (3.1) inequality holds.

The condition $\operatorname{Re} \dfrac{z[I^n_{NS} f(z)]'}{I^n_{NS} f(z)} > \alpha$ is equivalent to

$$\alpha - \operatorname{Re} \Big(\frac{z[I^n_{NS} f(z)]'}{I^n_{NS} f(z)} - 1 \Big) < 1.$$

We have

$$\alpha - \operatorname{Re} \Big(\frac{z[I^n_{NS} f(z)]'}{I^n_{NS} f(z)} - 1 \Big) \leq \alpha + \Big| \frac{z[I^n_{NS} f(z)]'}{I^n_{NS} f(z)} - 1 \Big|$$

$$= \alpha + \left| \frac{\sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)} z^j - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} z^j}{z - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)} z^j} \right| = \alpha + \left| \frac{\sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} z^{j-1} \big[\frac{1}{j} - 1 \big]}{1 - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)} z^{j-1}} \right|$$

$$\leq \alpha + \frac{\sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} |z|^{j-1} \big| \frac{1}{j} - 1 \big|}{1 - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)} |z|^{j-1}} < \alpha + \frac{\sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n,j)} \big[1 - \frac{1}{j} \big]}{1 - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)}}$$

$$= \frac{\alpha + \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{1}{j} - \frac{\alpha}{j}\right]}{1 - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)}}.$$

To finish our proof, we need to show

$$\frac{\alpha + \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{1}{j} - \frac{\alpha}{j}\right]}{1 - \sum\limits_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)}} < 1. \tag{3.3}$$

The (3.3) inequality is equivalent to

$$\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{\alpha}{j}\right] < 1 - \alpha, \tag{3.4}$$

which is the (3.1) condition. □

Let $E_{NS}(\alpha)$ be a subclass of $C_{NS}(\alpha)$. The class is defined as follows:

$$E_{NS}(\alpha) = \left\{ f \in T : \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha \text{ and } \alpha \in \left(0, \frac{1}{2}\right) \right\}. \tag{3.5}$$

Theorem 3.2. *Let $f \in T$ of the form (1.1). If $f \in E_{NS}(\alpha)$, then* $\operatorname{Re} \frac{I_{NS}^n f(z)}{z} > 0$.

Proof. Suppose $f \in E_{NS}(\alpha)$. Then

$$\left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha. \tag{3.6}$$

Let

$$I_{NS}^n f(z) = zp(z). \tag{3.7}$$

Differentiate (3.7), we obtain

$$[I_{NS}^n f(z)]' = zp'(z) + p(z). \tag{3.8}$$

Then (3.6) is equivalent to

$$\left| \frac{zp'(z)}{p(z)} \right| < 1 - 2\alpha.$$

If the condition $\operatorname{Re} p(z) = \operatorname{Re} \frac{I_{NS}^n f(z)}{z} > 0$ does not hold, then according to Lemma 2.2, there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$p(z_0) = ix$$

and

$$z_0 p'(z_0) = y \le -\frac{1+x^2}{2}.$$

These inequalities imply

$$\left|\frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|\frac{y}{ix}\right| \ge \left|\frac{\frac{1}{2}(1+x^2)}{x}\right| = \left|\frac{1}{2}\left(x+\frac{1}{x}\right)\right| \ge 1-2\alpha.$$

The above inequality contradicts (3.6) and consequently

$$\operatorname{Re} p(z) = \operatorname{Re}\frac{I^n_{NS} f(z)}{z} > 0,$$

where $z \in U$. □

Theorem 3.3. *Let*

$$F(z) = I_c f(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt, c \in \mathbb{N}.$$

If $f \in C_{NS}(\alpha)$, then $F = I_c(f) \in C_{NS}(\beta)$, where

$$\beta = \beta(\alpha, 2) = 1 - \frac{(1-\alpha)(c+1)^2}{(c+2)^2(2-\alpha)-(c+1)^2(1-\alpha)} \tag{3.9}$$

and $\beta > \alpha$, $\alpha \in [0,1)$.

Proof. Suppose $f \in C_{NS}(\alpha)$. Then by Theorem 3.1 we have

$$\sum_{j=2}^{\infty} \frac{a_j^2(j-\alpha)}{j^n C(n,j)(1-\alpha)} < 1.$$

We know that if f has the form (1.1), then

$$F(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt = z - \sum_{j=2}^{\infty}\frac{c+1}{c+j}a_j z^j,$$

and to prove that $F \in C_{NS}(\beta)$ is sufficient to have

$$\sum_{j=2}^{\infty}\frac{j-\beta}{j^n C(n,j)(1-\beta)}\left(\frac{c+1}{c+j}\right)^2 a_j^2 < 1.$$

This last inequality is implied by

$$\frac{j-\beta}{1-\beta}\cdot\frac{(c+1)^2 a_j^2}{j^n C(n,j)(c+j)^2} \le \frac{j-\alpha}{1-\alpha}\cdot\frac{a_j^2}{j^n C(n,j)}, \tag{3.10}$$

for all $j \in \mathbb{N}$ and $j \geq 2$.

From (3.10) we deduce that

$$\beta \leq 1 - \frac{(1-\alpha)(c+1)^2(j-1)}{(c+j)^2(j-\alpha)-(c+1)^2(1-\alpha)} = \beta(\alpha, j),$$

$j \in \mathbb{N}$, $j \geq 2$. We will prove that

$$\beta(\alpha, j) \geq \beta(\alpha, 2), \ j \in \mathbb{N}, \ j \geq 2.$$

Let consider the function $\varphi : [2, \infty) \to \mathbb{R}$,

$$\varphi(x) = \frac{x-1}{(x+c)^2(x-\alpha)-(c+1)^2(1-\alpha)}, \ x \in [2, \infty).$$

Then

$$\varphi'(x) = \frac{g(x)}{[(x+c)^2(x-\alpha)-(c+1)^2(1-\alpha)]^2},$$

where $g(x) = -2x^3 + (3-2c-\alpha)x^2 + (4c-2\alpha)x - 2c - (1-\alpha)$.

We have

$$g'(x) = -6x^2 + 2(3-2c-\alpha)x + 4c - 2\alpha,$$

$$g''(x) = -12x + 6 - 4c - 2\alpha < 0,$$

$x \in [2, \infty)$. Then

$$g'(x) \leq g'(2) = -12 - 4c - 6\alpha < 0, \ x \in [2, \infty)$$

and

$$g(x) \leq g(2) = -4 - 8\alpha - 2c - (1-\alpha) < 0, \ x \in [2, \infty).$$

We obtain $\varphi'(x) < 0$, $x \in [2, \infty)$ and from this

$$\beta(\alpha, j) = 1 - \varphi(j)(1-\alpha)(c+1)^2 \geq 1 - \varphi(2)(1-\alpha)(c+1)^2 = \beta(\alpha, 2,)$$

where $\beta(\alpha, 2)$ is given by (3.9). Finally $\beta > \alpha$ is equivalent to

$$1 - \alpha > \frac{(1-\alpha)(c+1)^2}{(c+2)^2(2-\alpha)-(c+1)^2(1-\alpha)}.$$

□

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Olga Engel - corresponding author
email: engel_olga@hotmail.com
Department of Mathematics,
Babeş - Bolyai University,
Cluj - Napoca,
Romania.

Yao Liang Chung,
email: chungyaoliang@gmail.com
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang,
Malaysia.

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