Journal of
Mathematics
and Applications

JMA No 39, pp 91-104 (2016)

On Some L_r-Biharmonic Euclidean Hypersurfaces

Akram Mohammadpouri and Firooz Pashaie

Abstract: In decade eighty, Bang-Yen Chen introduced the concept of biharmonic hypersurface in the Euclidean space. An isometrically immersed hypersurface $x : M^n \to \mathbb{E}^{n+1}$ is said to be biharmonic if $\Delta^2 x = 0$, where Δ is the Laplace operator. We study the L_r-biharmonic hypersurfaces as a generalization of biharmonic ones, where L_r is the linearized operator of the $(r+1)$th mean curvature of the hypersurface and in special case we have $L_0 = \Delta$. We prove that L_r-biharmonic hypersurface of L_r-finite type and also L_r-biharmonic hypersurface with at most two distinct principal curvatures in Euclidean spaces are r-minimal.

AMS Subject Classification: Primary: 53-02, 53C40, 53C42; Secondary 58G25.
Keywords and Phrases: Linearized operator L_r; L_r-biharmonic hypersurfaces; L_r-finite type hypersurfaces; r-minimal.

1. Introduction

The concept of biharmonic surfaces in Euclidean space has applications in elasticity and fluid mechanics. In sixty decade, G.B. Airy and J.C. Maxwell have studied the plane elastic problems in terms of the biharmonic equation ([1, 13]). In more general case, the subject of polyharmonic functions was developed by E. Almansi, T. Levi-Civita, M. Nicolaescu. In addition to the differential geometric point of view, biharmonic maps are appeared in PDE theory as solutions of a fourth order strongly elliptic semilinear PDE and in computational geometry as the biharmonic Bezier surfaces.

Clearly, the importance of biharmonic maps will be serious where harmonic maps do not exist. For example, since there exists no harmonic map as $\mathbb{T}^2 \to \mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1, it is important to find a biharmonic map from $\mathbb{T}^2$ into $\mathbb{S}^2$ (see in [9]). Obviously, harmonic maps are biharmonic but not vis versa. Biharmonic non-harmonic maps are called *proper-biharmonic*. The variational problem associated to the bienergy functional on the set of Riemannian metrics on a domain gave rise to the biharmonic stress-energy tensor. This is useful to obtain a new example of proper-biharmonic maps for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds.

A differential geometric motivation of the subject of biharmonic hypersurfaces is the well-known conjecture of Bang-Yen Chen (in 1987) which says that the biharmonic surfaces in Euclidean 3-spaces are minimal ones. Later on, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([8]). Also, in 1995, Hasanis and Vlachos proved extended Chen's result to the hypersurfaces in Euclidean 4-spaces ([10]). Under the assumption of completeness, Akutagawa and Maeta ([2]) gave a generalization of the result to the global version of Chen's conjecture on biharmonic submanifolds in Euclidean spaces. On the other hand, Dimitrić has found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen and also L.J. Alias, S.M.B. Kashani and others. One can see main results in the last chapter of Chen's book ([6]). In [11], Kashani has introduced the notion of L_r-finite type hypersurfaces as an extension of finite type ones in the Euclidean space, which is followed in the doctoral thesis of first author.

The map L_r, as an extension of the Laplacian operator $L_0 = \Delta$, stands for the linearized operator of the first variation of the $(r+1)$th mean curvature of the hypersurface (see, for instance, [17]). This operator is given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the rth Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f.

It seems interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_r. We call these hypersurfaces *L_r-biharmonic*. Since r-minimal immersions are L_r-biharmonic, one can ask naturally "what about the vise versa?"

In this paper, we study L_r-biharmonic hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$. Recently, Aminian and Kashani proved ([5]) the L_r-conjecture for the hypersurfaces with at most two distinst prinicipal curvatures. In this paper, we give an alternative proof of this result by a different method. As our first result on L_r-biharmonic hypersurfaces, we prove that each L_r-biharmonic hypersurface of L_r-finite type in the Euclidean space is r-minimal. Then, we show that any L_r-biharmonic hypersurface in Euclidean space with at most two distinct principal curvatures is r-minimal. The case $r = 0$ (biharmonic hypersurfaces) was studied by Dimitrić, [7]. He proved that, biharmonic hypersurface of finite type or concerning at most two distinct principal curvatures is minimal.

Here are our main results.

Theorem 1.1. *The L_r-biharmonic hypersurfaces of L_r-finite type in Euclidean spaces are r-minimal.*

Theorem 1.2. *The only L_r-biharmonic hypersurfaces of Euclidean spaces $\mathbb{E}^{n+1}$ with at most two distinct principal curvatures are the r-minimal ones* $(0 \leq r \leq n-1)$.

Corollary 1.3. Every L_1-biharmonic surface in $\mathbb{E}^3$ is flat.

Corollary 1.4. Let M^n be a conformally flat L_r-biharmonic hypersurface of $\mathbb{E}^{n+1}$, $n > 3$. Then M^n is r-minimal.

After the preliminaries in section 2, in the third section, we prove the main results.

2. Preliminaries

In this section, we introduce some basic notations and facts that will appear along the paper from [19], [4] and [11].

Consider an isometrically immersed hypersurface $x : M^n \to \mathbb{E}^{n+1}$ in the Euclidean space. We choose a local orthonormal frame $\{e_A\}_{1\leq A\leq n+1}$ in $\mathbb{E}^{n+1}$, with dual coframe $\{\omega_A\}_{1\leq A\leq n+1}$, such that, at each point of M, $e_1, \ldots, e_n$ are tangent to M and e_{n+1} is the positively oriented unit normal vector. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, ..., \leq n+1; \quad 1 \leq i, j, k, ..., \leq n.$$

Then the structure equations of $\mathbb{E}^{n+1}$ are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB}. \tag{2}$$

When restricted to M, we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1i} \wedge \omega_i. \tag{3}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{n+1i} = \sum_{j=1}^{n} h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{4}$$

This gives the second fundamental form of M, as $B = \sum\limits_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature H is defined by $H = \frac{1}{n}\sum\limits_{i} h_{ii}$. From (1) - (4) we obtain the structure equations of M (see [19]).

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{5}$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l, \tag{6}$$

and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{7}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{8}$$

Thus, by exterior differentiation of (4), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{9}$$

We choose $e_1, \ldots, e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{10}$$

The rth mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{r} H_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r}. \tag{11}$$

And $H_n = \lambda_1 \cdots \lambda_n$, is called the Gauss-Kronecker curvature of M. A hypersurface with zero $(r+1)$th mean curvature in $\mathbb{R}^{n+1}$ is called r-minimal. To get more information about r-minimal Euclidean hypersurfaces, the reader is referred to [3, 20].

The classical Newton transformations $P_r : \chi(M) \to \chi(M)$ are defined inductively by the shape operator S as

$$P_0 = I \quad \text{and} \quad P_r = \binom{n}{r} H_r I - S \circ P_{r-1},$$

for every $r = 1, \ldots, n$, where I denotes the identity transformation in $\chi(M)$. Equivalently,

$$P_r = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j.$$

Note that by the Cayley-Hamilton theorem stating that any operator is annihilated by its characteristic polynomial, we have $P_n = 0$.

Since each $P_r(p)$ is also a self-adjoint linear operator on each tangent plane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized: if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{i_1 < \cdots < i_r, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p), \tag{12}$$

for every $1 \le i \le n$. We have the following formulae for the Newton transformations, [4].

$$tr(P_r) = c_r H_r, \tag{13}$$

$$tr(S \circ P_r) = c_r H_{r+1}, \tag{14}$$

and

$$tr(S^2 \circ P_{n-1}) = nH_1H_n, \quad tr(S^2 \circ P_r) = \binom{n}{r+1}(nH_1H_{r+1} - (n-r-1)H_{r+2}) \tag{15}$$

for $r = 1, \ldots, n-2$, where

$$c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}.$$

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$$L_r(f) = tr(P_r \circ \nabla^2 f).$$

Here $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$< \nabla^2 f(X), Y >=< \nabla_X(\nabla f), Y >,$$

where $X, Y \in \chi(M)$, ∇f is the gradient of f and ∇ is the Levi-Civita connections on M.

Now we recall the definition of an L_r-finite type hypersurface from [11], which is the basic notion of the paper.

Definition 2.1. An isometrically immersed hypersurfaces $x : M^n \to \mathbb{E}^{n+1}$ is said to be of L_r-finite type if x has a finite decomposition $x = \sum_{i=0}^m x_i$, for some positive integer m satisfying the condition that $L_r x_i = \kappa_i x_i, \kappa_i \in \mathbb{R}, 1 \leq i \leq m$, where $x_i : M^n \to \mathbb{E}^{n+1}$ are smooth maps, $1 \leq i \leq m$, and x_0 is constant. If all κ_i's are mutually different, M^n is said to be of L_r-m-type. An L_r-m-type hypersurface is said to be null if some $\kappa_i; 1 \leq i \leq m$, is zero.

3. L_r-biharmonic hypersurfeces

Consider $x : M^n \to \mathbb{E}^{n+1}$ a connected orientable hypersurface immersed into the Euclidean space, with the Gauss map N. Then M^n is called a L_r-biharmonic hypersurface if and only if $L_r^2 x = 0$ or equivalently, $L_r(H_{r+1}N) = 0$ (see [4]).

By definition of the L_r-biharmonic hypersurface, it is clear that r-minimal immersions are trivially L_r-biharmonic. By using formula for $L_r^2 x$ of [4] and the considering normal and tangent parts of the L_r-biharmonic condition $L_r^2 x = 0$, one obtains necessary and sufficient conditions for M^n to be L_r-biharmonic in $\mathbb{E}^{n+1}$, namely

$$L_r H_{r+1} = \binom{n}{r+1} H_{r+1}(nH_1H_{r+1} - (n-r-1)H_{r+2}) = tr(S^2 \circ P_r)H_{r+1} \tag{16}$$

and
$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2}\binom{n}{r+1} H_{r+1} \nabla H_{r+1}. \tag{17}$$

In [7], Dimitrić proved that each biharmonic hypersurface of finite type in a Euclidean space is minimal. In Theorem 1.1, we follow Dimitrić's work and prove that each L_r-biharmonic hypersurface of L_r-finite type in a Euclidean space is r-minimal. Case $r = 0$ corresponds to the classical one.

3.1. Proof of Theorem 1.1

Proof. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometrically immersed L_r-biharmonic hypersurface of L_r-finite type in the Euclidean space. Then it has finite decomposition
$$x = x_0 + x_{t_1} + \cdots + x_{t_k}, \tag{18}$$
with $L_r x_0 = 0,\ L_r x_{t_i} = \lambda_{t_i} x_{t_i}$ for nonzero distinct eigenvalues $\lambda_{t_1}, \ldots, \lambda_{t_k}$ of L_r. By taking L_r^s of (18) we find
$$0 = L_r^s x = \lambda_{t_1}^s x_{t_1} + \cdots + \lambda_{t_k}^s x_{t_k}, \qquad s = 2, 3, \ldots. \tag{19}$$
Since $\lambda_{t_1}, \ldots, \lambda_{t_k}$ are distinct eigenvalues of L_r, system (19) is incinsistent unless $k = 0$. Thus, $x = x_0$, which implies that M is r-minimal. □

In [6], Chen proved that every biharmonic surface in $\mathbb{E}^3$ is minimal. Dimitrić ([7]) generalizing Chen's result, proved that any biharmonic hypersurface with at most two distinct principal curvatures is minimal. In Theorem 1.2, we generalize this result and prove that any L_r-biharmonic Euclidean hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{n+1}$ is r-minimal.

Since always exists an open dense subset of M on which the multiplicities of the principal curvatures are locally constant (see Reckziegel [16]), therefore we use the following Lemma locally for the proof of Theorem 1.2.

Lemma 3.1. [15] *Let M be an n-dimensional hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ such that multiplicities of principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than one, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

3.2. Proof of Theorem 1.2

Proof. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometrically immersed L_r-biharmonic Euclidean hypersurface. It is enough to prove that $\mathcal{U} = \{p \in M : \nabla H_{r+1}^2(p) \neq 0\}$, our objective is to show that $\mathcal{U}$ is empty.

In order to prove the Theorem 1.2, we considering three different cases as follows.

Case I: $r = n - 1$.
Case II: $r \neq n - 1$ and the multiplicities are greater than one.
Case III: $r \neq n - 1$ and one of the principal curvatures is simple.

Case I: First, we show that the Gauss-Kronecker curvature of M is constant. By using formulae (16) and (17) on $\mathcal{U}$ we get

$$(SoP_{n-1})\nabla H_n = -\frac{1}{2} H_n \nabla H_n, \tag{20}$$

$$L_{n-1} H_n = n H_1 H_n^2. \tag{21}$$

But by the Cayley-Hamilton theorem we have $P_n = 0$, so

$$SoP_{n-1} = H_n I, \quad (SoP_{n-1})\nabla H_n = H_n \nabla H_n,$$

which jointly with (20) yields $\nabla H_n^2 = 0$ on $\mathcal{U}$, which is a contradiction.

If $H_n \neq 0$, by using (21) we obtain that the mean curvature is constant. By the fact that M has at most two principal curvatures and H, H_n are constant, we get that the principal curvatures are constant, so M is isoparametric. A classical result of B. Segre [18], states that isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ with non zero Gauss-Kronecker curvature are locally isometric to S^n. On the other hand, since S^n is of L_{n-1}-1-type (see [11]), by using Theorem 1.1, we conclude that it is impossible. This finishes the proof of case I.

Case II: Since S^n is of L_{n-1}-1-type (see [11]), therefore, if M^n is totally umbilical, then M^n is a piece of $\mathbb{E}^n$. Therefore, we assume that M has two distinct principal curvatures of multipilicities q and $n - q$, $(q\ , n - q > 1)$.

Consider $\{e_1, \ldots, e_n\}$, to be a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for every $i = 1, \ldots, n$. We assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_q = \lambda, \qquad \lambda_{q+1} = \cdots = \lambda_n = \mu.$$

Therefore from (12) we have

$$P_{r+1} e_i = \mu_{i,r+1} e_i,$$

with

$$\mu_{i,r+1} = \sum_{i_1 < \cdots < i_{r+1}, i_j \neq i} \lambda_{i_1} \ldots \lambda_{i_{r+1}}.$$

So, we get

$$\begin{aligned} \mu_{1,r+1} = \cdots = \mu_{q,r+1} &= \sum_s \binom{q-1}{s} \binom{n-q}{r+1-s} \lambda^s \mu^{r+1-s}, \\ \mu_{q+1,r+1} = \cdots = \mu_{n,r+1} &= \sum_s \binom{q}{s} \binom{n-q-1}{r+1-s} \lambda^s \mu^{r+1-s}. \end{aligned} \tag{22}$$

We obtain from (11) that

$$\binom{n}{r+1}H_{r+1}=\sum_s\binom{q}{s}\binom{n-q}{r+1-s}\lambda^s\mu^{r+1-s}. \tag{23}$$

Since $r\neq n-1$, it follows from the inductive definition of P_{r+1} that (17) is equivalent to

$$P_{r+1}(\nabla H_{r+1}^2)=\frac{3}{2}\binom{n}{r+1}H_{r+1}\nabla H_{r+1}^2 \qquad \text{on } \mathcal{U}. \tag{24}$$

Therefore, writing

$$\nabla H_{r+1}^2=\sum_{i=1}^{n}<\nabla H_{r+1}^2,e_i>e_i, \tag{25}$$

we see that (24) is equivalent to

$$<\nabla H_{r+1}^2,e_i>(\mu_{i,r+1}-\frac{3}{2}\binom{n}{r+1}H_{r+1})=0 \quad \text{on } \mathcal{U},$$

for every $i=1,\dots,n$. So, there is no loss of generality, assuming that,

$$\mu_{1,r+1}=\cdots=\mu_{q,r+1}=\frac{3}{2}\binom{n}{r+1}H_{r+1}. \tag{26}$$

Let us denote the integral submanifolds through $x\in\mathcal{U}$ corresponding to λ and μ by $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$ respectively. From Lemma 3.1, we know that λ is constant on $\mathcal{U}_1^q(x)$. (22), (23) and (26) imply that μ is constant on $\mathcal{U}_1^q(x)$. Again by Lemma 3.1, we get that μ is constant on $\mathcal{U}_1^{n-q}(x)$. It now follows from [12], p. 182, Vol. I, that $\mathcal{U}$ is locally isometric to the Riemannian product of the maximal integral manifolds $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$. Therefore, μ is constant on $\mathcal{U}$. By the same assertion, we know that λ is constant on $\mathcal{U}$, so H_{r+1} is constant on $\mathcal{U}$, which is a contradiction. Hence H_{r+1} is constant on M. If $H_{r+1}\neq 0$, then from (16), we obtain that $tr(S^2\circ P_r)$ is constant. By the fact that M has two principal curvatures and H_{r+1}, $tr(S^2\circ P_r)$ are constant, we get that the principal curvatures are constant. So, M is isoparametric. The discussion as in the last part of the proof of case I, we get the result in Case II.

Case III: In this case, we suppose that M has two distinct principal curvatures of multiplicities 1 and $n-1$. Assume that $\mathcal{U}\neq\emptyset$ (then we will try to get a contradiction). One can express H_{r+1} as a polynomial in λ (the non simple principal curvature of M) with constant coefficients, after that we express λ as a constant multiple of the simple principal curvature of M. By using Otsuki's Lemma (Lemma 3.1), the structure equations of M, and the fact that M is L_r-biharmonic hypersurface, we get that λ satisfies a polynomial with constant coefficients. So λ is constant, hence H_{r+1} is constant, a contradiction with $\mathcal{U}\neq\emptyset$. Therefore, $\mathcal{U}$ is empty.

Here, is the detailed treatment of the proof.

With the assumption that $\mathcal{U}\neq\emptyset$, consider $\{e_1,\dots,e_n\}$, to be a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for every $i = 1, \ldots, n$. We assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \qquad \lambda_n = \mu.$$

Therefore we have

$$\begin{aligned} &\mu_{1,r+1} = \cdots = \mu_{n-1,r+1} = \binom{n-2}{r+1}\lambda^{r+1} + \binom{n-2}{r}\lambda^r \mu, \\ &\mu_{n,r+1} = \binom{n-1}{r+1}\lambda^{r+1}. \end{aligned} \tag{27}$$

We obtain from (12) that

$$\binom{n}{r+1} H_{r+1} = \binom{n-1}{r+1}\lambda^{r+1} + \binom{n-1}{r}\lambda^r \mu. \tag{28}$$

Since $r \neq n-1$, it follows from the inductive definition of P_{r+1} that (17) is equivalent to

$$P_{r+1}(\nabla H_{r+1}^2) = \frac{3}{2}\binom{n}{r+1} H_{r+1} \nabla H_{r+1}^2 \qquad \text{on } \mathcal{U}. \tag{29}$$

Therefore, by the formula

$$\nabla H_{r+1}^2 = \sum_{i=1}^{n} < \nabla H_{r+1}^2, e_i > e_i, \tag{30}$$

we see that (29) is equivalent to

$$< \nabla H_{r+1}^2, e_i > (\mu_{i,r+1} - \frac{3}{2}\binom{n}{r+1} H_{r+1}) = 0 \quad \text{on } \mathcal{U},$$

for every $i = 1, \ldots, n$. Hence, for every i such that $< \nabla H_{r+1}^2, e_i > \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,r+1} = \frac{3}{2}\binom{n}{r+1} H_{r+1}. \tag{31}$$

So for the expression ∇H_{r+1}^2 in (30) we consider two subcases.

Subcases 1. $< \nabla H_{r+1}^2, e_n > \neq 0$, by using (27) and (31), we obtain that

$$H_{r+1} = \frac{2}{3}\frac{(n-r-1)}{n}\lambda^{r+1}. \tag{32}$$

Subcases 2. $< \nabla H_{r+1}^2, e_n >= 0$, so on $\mathcal{U}$ we have $< \nabla H_{r+1}^2, e_j > \neq 0$ for some $j = 1, \ldots, n-1$. By using (27), (31) and the formula of $tr(P_{r+1})$, we obtain that

$$H_{r+1} = \frac{(n-r-1)}{n(-\frac{1}{2}n - r + \frac{1}{2})}\lambda^{r+1}. \tag{33}$$

Both states requires the same calculation, so, we consider just state I.

By Lemma 3.1, let us denote the maximal integral submanifold through $x \in \mathcal{U}$, corresponding to λ by $\mathcal{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_i \lambda_i \omega_i \qquad d\mu = \sum_j \mu_j \omega_j. \tag{34}$$

Then Lemma 3.1 implies that $\lambda_1 = \cdots = \lambda_{n-1} = 0$. We can assume that $\lambda > 0$ on $\mathcal{U}$, then (28) and (32) yields

$$\mu = \frac{r+1-n}{3r+3}\lambda. \tag{35}$$

By means of (8) and (10), we obtain that

$$\sum_k h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \tag{36}$$

We adopt the notational convention that $1 \leq a, b, c, \ldots \leq n-1$.

From (34) and (36), we have

$$\begin{aligned} &h_{ijk} = 0, \quad \text{if } i \neq j, \quad \lambda_i = \lambda_j, \\ &h_{aab} = 0, \qquad h_{aan} = \lambda_n, \\ &h_{nna} = 0, \qquad h_{nnn} = \mu_n. \end{aligned} \tag{37}$$

Combining this with (9) and the formula

$$\sum_i h_{ani}\omega_i = dh_{an} + \sum_i h_{in}\omega_{ia} + \sum_i h_{ai}\omega_{in} = (\lambda - \mu)\omega_{an},$$

we obtain from (35)

$$\omega_{an} = \frac{\lambda_n}{\lambda - \mu}\omega_a = \frac{(3r+3)\lambda_n}{(2r+2+n)\lambda}\omega_a. \tag{38}$$

Therefore, we have

$$d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ. We may put $\omega_n = ds$.

Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_n ds, \qquad \lambda_n = \lambda'(s),$$

so, from (38), we get

$$\omega_{an} = \frac{\lambda_n}{\lambda - \mu}\omega_a = \frac{(3r+3)\lambda'(s)}{(2r+2+n)\lambda}\omega_a. \tag{39}$$

According to the structure equations of $\mathbb{E}^{n+1}$ and (39), we may compute

$$\begin{aligned} d\omega_{an} &= \sum_{b=1}^{n-1}\omega_{ab}\wedge\omega_{bn}+\omega_{an+1}\wedge\omega_{n+1n} \\ &= \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)\sum_{b=1}^{n-1}\omega_{ab}\wedge\omega_b-\lambda\mu\omega_a\wedge ds, \\ d\omega_{an} &= d\left\{\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\omega_a\right\} \\ &= \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)' ds\wedge\omega_a+\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)d\omega_a \\ &= \left\{-\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)'+\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)^2\right\}\omega_a\wedge ds \\ &+\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)\sum_{b=1}^{n-1}\omega_{ab}\wedge\omega_b. \end{aligned} \tag{40}$$

Then we obtain from two equalities above that

$$\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)'-\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)^2-\lambda\mu=0. \tag{41}$$

Combining (41) with (35), we have

$$\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)'-\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda}\right)^2-\left(\frac{r+1-n}{3r+3}\right)\lambda^2=0. \tag{42}$$

Let us define a function $\beta(s)$, $s\in(-\infty,+\infty)$ by $\beta=\left(\frac{1}{\lambda}\right)^{\frac{3r+3}{2r+2+n}}$, then (42) reduces to

$$\beta''=\left(\frac{n-r-1}{3r+3}\right)\beta^{\frac{-r-1-2n}{3r+3}}. \tag{43}$$

Integrating (43), we obtain

$$(\beta')^2=-\beta^{\frac{2r+2-2n}{3r+3}}+c, \tag{44}$$

where c is the constant of integration.
(44) is equivalent to

$$(\lambda')^2=-\left(\frac{2+2r+n}{3r+3}\right)^2\lambda^{\frac{8r+4n+8}{2r+2+n}}+c\left(\frac{2+2r+n}{3r+3}\right)^2\lambda^{\frac{10r+10+2n}{2r+2+n}}. \tag{45}$$

Now by the definition of $L_rH_{k+1}=tr(P_r\circ\nabla^2H_{r+1})$, we compute L_rH_{r+1}. So we need to compute $\nabla_{e_a}\nabla H_{r+1},\ \nabla_{e_n}\nabla H_{r+1},\ P_r(e_a)$ and $P_r(e_n)$.
From (32) we have

$$\nabla H_{r+1}=\frac{2(r+1)(n-r-1)}{3n}\lambda^r\lambda' e_n. \tag{46}$$

By using (39) and (46) we obtain

$$\begin{aligned}\nabla_{e_a}\nabla H_{r+1} &= \frac{2(r+1)(n-r-1)}{3n}\lambda^r\lambda'\nabla_{e_a}e_n = \frac{2(r+1)(n-r-1)}{3n}\lambda^r\lambda'\sum_b\omega_{nb}(e_a)e_b \\ &= -\frac{2(n-r-1)(r+1)^2}{n(2r+2+n)}\lambda^{r-1}\lambda'^2 e_a \\ \nabla_{e_n}\nabla H_{r+1} &= \frac{2(r+1)(n-r-1)}{3n}\nabla_{e_n}(\lambda^r\lambda' e_n) \\ &= \frac{2r(r+1)(n-r-1)}{3n}\lambda^{r-1}\lambda'^2 e_n + \frac{2(r+1)(n-r-1)}{3n}\lambda^r\lambda'' e_n.\end{aligned} \tag{47}$$

By using (27) and (35), we compute $P_r(e_a)$ and $P_r(e_n)$.

$$\begin{aligned}P_r(e_a) &= \mu_{a,r}e_a = \left(\sum_{i_1<\cdots<i_r, i_j\neq a}\lambda_{i_1}\ldots\lambda_{i_r}\right)e_a = \binom{n-2}{r}\frac{2r+3}{3r+3}\lambda^r e_a, \\ P_r(e_n) &= \binom{n-1}{r}\lambda^r e_n.\end{aligned} \tag{48}$$

From (47) and (48), we get

$$\begin{aligned}L_rH_{r+1} = c_rH_{r+1}\Bigg(&\frac{(-2r-3)(r+1)(n-r-1)}{n(2r+2+n)}\lambda^{r-2}\lambda'^2 \\ &+ \frac{r(r+1)}{n}\lambda^{r-2}\lambda'^2 + \frac{r+1}{n}\lambda^{r-1}\lambda''\Bigg).\end{aligned} \tag{49}$$

Since M^n is of L_r-biharmonic hypersurface, hence from (16), we get

$$L_rH_{r+1} = H_{r+1}tr(S^2\circ P_r) = H_{r+1}\binom{n-1}{r}\frac{2nr+3n-2r-2r^2}{3r+3}\lambda^{r+2}. \tag{50}$$

Combining (49) and (50), we have

$$\lambda\lambda'' + \left(r + \frac{(-2r-3)(n-r-1)}{2r+2+n}\right)\lambda'^2 - \binom{n-1}{r}\frac{n(2nr+3n-2r-2r^2)}{(r+1)(3r+3)}\lambda^4 = 0. \tag{51}$$

(42) is equivalent to

$$\lambda\lambda'' = \frac{5r+5+n}{2r+2+n}\lambda'^2 + \frac{(2r+2+n)(r+1-n)}{(3r+3)^2}\lambda^4. \tag{52}$$

Thus, putting together (51) and (52) one has

$$\begin{aligned}&\frac{4r^2+12r-rn-2n+8}{2r+2+n}\lambda'^2 \\ &+ \frac{(2r+2+n)(r+1-n)+3\binom{n-1}{r}n(2nr+3n-2r-2r^2)}{(3r+3)^2}\lambda^4 = 0.\end{aligned} \tag{53}$$

We deduce, using (45), (53) and (32), that H_{r+1} is locally constant on $\mathcal{U}$, which is a contradiction with the definition of $\mathcal{U}$. Hence H_{r+1} is constant on M. The discussion as in the last part of the proof of the case I, we get the result. □

An important consequence of the Theorem is the classification of conformally flat L_r-biharmonic hypersurfaces M^n for $n > 3$.

The dimension of the hypersurface plays an important role in the study of conformally flat Euclidean hypersurfaces. For $n = 2$, the existence of isothermal coordinates means that any Riemannian surface is conformally flat. For $n > 3$, the result of Cartan-Schouten states that a conformally flat hypersurface is characterized with two principal curvatures that one multiplicity at least $n-1$ (see [14] for more details). This significant fact is crucial in our classification of L_r-biharmonic conformally flat Euclidean hypersurfaces M^n for $n > 3$.

As a simple consequence of Theorem 1.2; case III, we obtain the Corollary 1.4.

References

[1] G.B. Airy, *On the strains in the interior of beams*, Philos. Trans. R. Soc. London Ser. A 153 (1863) 49-79.

[2] K. Akutagawa, S. Maeta, *Biharmonic properly immersed submanifolds in euclidean spaces*, Geom. Dedicata 164 (2013) 351-355.

[3] H. Alencar, M. Batista, *Hypersurfaces with null higher order mean curvature*, Bull. Braz. Math. Soc., New Series 41(4) (2010) 481-493.

[4] L.J. Alías, N. Gürbüz, *An Extension of takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata 121 (2006) 113-127.

[5] M. Aminian, S.M.B. Kashani, *L_k-biharmonic hypersurfaces in euclidean space*, Taiwan J. Math., DOI: 10.11650/tjm.18.2014.4830, (2014) 113-127.

[6] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, Series in Pure Mathematics, 2. World Scientific Publishing Co, Singapore, 2014.

[7] I. Dimitrić, *Quadric Representation and Submanifolds of Finite Type*, Doctoral Thesis, Michigan State University, 1989.

[8] I. Dimitrić, *Submanifolds of E^n with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sin. 20 (1992) 53-65.

[9] J. Eells, J.C. Wood, *Restrictions on harmonic maps of surfaces*, Topology 15 (1976) 263-266.

[10] T. Hasanis, T. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr. 172 (1995) 145-169.

[11] S.M.B. Kashani, *On some L_1-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$*, Bull. Korean Math. Soc. 46, 1 (2009) 35-43.

[12] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley-Interscience, New York, (1963) Vol. II, (1969).

[13] J.C. Maxwell, *On reciprocal diagrams in space, and their relation to Airy's function of stress*, Proc. London. Math. Soc. 2 (1868) 102.

[14] S. Nishikawa, Y. Maeda, *Conformally flat hypersurfaces in a conformally at Riemannian manifold*, Tohoku Math. J. 26 (1974) 159-168.

[15] T. Otsuki, *Minimal Hypersurfaces in a Riemannian Manifold of Constant Curvature*, Amer. J. Math. 92, 1 (1970) 145-173.

[16] H. Reckziegel, *On the eigenvalues of the shape operator of an isometric immersion into a space of constant curvature*. Math. Ann. 243 (1979) 71-82.

[17] R.C. Reilly, *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geom. 8, 3 (1973) 465-477.

[18] B. Segre, *Famiglie di ipersuperficie isoparametrische negli spazi euclidei ad un qualunque numero di dimensioni*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur 27 (1938) 203-207.

[19] G. Wei, *Complete hypersurfaces in a Eculidean space* $\mathbb{R}^{n+1}$ *with constant mth mean curvature*, Diff. Geom. Appl. 26 (2008) 298-306.

[20] B.G. Yang, X.M. Liu, *r-minimal hypersurfaces in space forms*, Journal of Geometry and Physics 59 (2009) 685-692.

DOI: 10.7862/rf.2016.7

Akram Mohammadpouri - corresponding author
email: pouri@tabrizu.ac.ir

Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz, Iran.

Department of Mathematics,
Faculty of Basic Sciences,
University of Maragheh,
P.O.Box 55181-83111, Maragheh, Iran.

Firooz Pashaie
email: f_pashaie@maragheh.ac.ir

Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz, Iran.

Department of Mathematics,
Faculty of Basic Sciences,
University of Maragheh,
P.O.Box 55181-83111, Maragheh, Iran.

Received 15.11.2014 *Accepted 7.05.2016*